Self-adjoint elliptic problems in domains with cylindrical ends under weak assumptions on the stabilization of coefficients

Kalvine V. O.

Abstract

The general self-adjoint elliptic boundary value problems are considered in a domain $G \subset \mathbb{R}^{n+1}$ with finitely many cylindrical ends. The coefficients are stabilizing (as $x \to \infty$, $x \in G$) so slowly that we can only describe some "structure" of solutions far from the origin. This problem may be understood as a model of "generalized branching waveguide." We introduce a notion of the energy flow through the cross-sections of the cylindrical ends and define outgoing and incoming "waves." An augmented scattering matrix is introduced. Analyzing the spectrum of this matrix one can find the number of linearly independent solutions to the homogeneous problem decreasing at infinity with a given rate. We discuss the statement of problem with so-called radiation conditions and enumerate self-adjoint extensions of the operator of the problem.

1 Introduction

In domain $G \subset \mathbb{R}^{n+1}$ with finitely many cylindrical ends we consider the general formally self-adjoint boundary value problem. The coefficients tend to limits (as $x \to \infty$, $x \in G$) too slow to allow obtaining an asymptotic of solution at infinity. Using the results of the paper [1] (see also [2, Section 8.5]), one can get some "structure" of solution to the problem: far from the origin a solution is represented as a linear combination of some functional series plus a remainder. The coefficients in the linear combination remain unknown. In this paper we develop an approach, which, in particular, allows

to derive expressions for the coefficients in the structure of solution to the problem under consideration.

Let $\Pi^r_+ = \{(y,t) : y \in \Omega^r, t > 0\}$, $r = 1, \dots, N$, stand for the cylindrical ends, where Ω^r is the cross-section. (The domain G coincides with the union $\Pi^1_+ \cup \dots \cup \Pi^N_+$ outside a large ball.) With every cylindrical end Π^r_+ we associate limit and model problems in the cylinder $\Pi^r = \{(y,t) : y \in \Omega^r, t \in \mathbb{R}\}$.

As the coefficients of limit problem we take the limits of coefficients of the original problem as $t \to +\infty$, $(y,t) \in \Pi^r_+$. It is assumed that the limit problems are elliptic. Since the operator of the original problem is formally self-adjoint, the operators of the limit problems are formally self-adjoint as well. As is known (see e.g. [3, Chapter 5]), one can consider every limit problem as a model of "generalized waveguide." This means that a generalized notion of the energy flow through the cross-section of the cylinder is introduced, the solution to the homogeneous problem is called incoming (outgoing) wave if the energy flow associated with the solution is positive (negative). The amplitudes of such waves may grow with power or even with exponential rate at infinity.

The operator of the model problem is formally self-adjoint and depends on the parameter $T \in \mathbb{R}$. The coefficients of the model problem coincide with the coefficients of the original problem on the set $\{(y,t) \in P^r_+, t > T+3\}$ and with their limits (as $t \to +\infty$, $(y,t) \in \Pi^r_+$) on the set $\{(y,t) \in \Pi^r, t < T\}$. The coefficients of the model problem tend to the coefficients of the limit one as $T \to +\infty$. Thus a solution to the homogeneous model problem can be obtained in the form of functional series by the method of successive approximations, as the first approximation it is natural to take a wave of the limit problem. On the analogy of the limit problem, for the model problem we introduce a notion of the energy flow through the cross-section of the cylinder. The formula for the energy flow through the cross-section $\{(y,t) \in \Pi^r, t = R\}$, $R < T$, is the same for both (limit and model) problems because the coefficients of the problems coincide on the set $\{(y,t) \in \Pi^r, t < T\}$. Moreover, it turns out that a wave and the correspondent solution to the homogeneous model problem have equal energy flows through the left infinitely distant cross-section of Π^r. This allows to calculate the energy flows of obtained solutions to the homogeneous model problem and also allows to separate these solutions into incoming and outgoing waves (of the model problem). Due to the formally self-adjointness of the model problem operator, the energy flows of such waves remain constant along the cylinder. Recall that the coefficients of the model problem coincide with the coefficients of original problem on

the set $\{(y,t) \in \Pi^r_+, t > T+3\}$. Owing to this fact, one can consider the domain G as a branching waveguide, where the waves obtained for the model problem in Π^r propagate along the cylindrical end Π^r_+ of G, $r = 1, \dots, N$. Using a modification of the scheme suggested in [1, Theorem 6.2], we get the structure of solutions to the problem in G: far from the origin a solution is represented as a linear combination of the waves plus a remainder. Some waves properties obtained on the previous step allow us to derive the formulas for the coefficients in the structure of solution. The results are represented in Theorem 3.10 and Theorem 3.11.

The remaining part of the paper basically contains corollaries of the theorems 3.10 and 3.11. We omit the proofs because they almost repeat the proofs of the similar assertions in [3, Chapther 5] or in [11], where it is assumed that the coefficients are stabilizing with exponential rate. The changes in the proofs mainly consist in usage of Theorem 3.10 or Theorem 3.11 instead of asymptotic representations. In the main text we insert the exact references to the needed proofs.

The operator of the problem acts in weighted spaces. We obtain some information about the kernel of the problem (Propositions 4.2 and 4.3) and introduce "scattering matrices." These unitary matrices take into account waves growing at infinity. Analyzing the spectrum of this matrix one can find the number of linearly independent solutions to the homogeneous problem decreasing at infinity with a given rate (cf. Proposition 4.5). We discuss the statement of problem with "radiation conditions:" the domain of operator contains only functions with prescribed structure at infinity. This is a way to choose a solution (with a certain arbitrariness) (cf. Propositions 4.6 and 4.7). The intrinsic radiation conditions (the solution mainly consists of outgoing waves) can be utilized in every case. To verify whether given radiation conditions can be used, it is required to know the scattering matrix (cf. Proposition 4.8). In Section 4.3 the self-adjoint extensions of operator of the problem are found.

Some of the results proved in our paper were announced earlier in the work [4].

2 Statement of the problem and preliminaries

2.1 Domain and self-adjoint boundary value problem

Let $G \subset \mathbb{R}^{n+1}$ be a domain with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi_+^1 \cup \cdots \cup \Pi_+^N$ of non-overlapping semicylinders; here $\Pi^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r > 0\}$, (y^r, t^r) are local coordinates, and the cross-section Ω^r is bounded domain in $\mathbb{R}^n$. In the domain G we introduce a formally self-adjoint $k \times k$-matrix $\mathcal{L}$ of differential operators $\mathcal{L}_{ij}(x, D_x)$ with smooth coefficients, where $\operatorname{ord} \mathcal{L}_{ij} = \tau_i + \tau_j$, the numbers $\tau_1, \ldots, \tau_k$ are non-negative integers, and $\tau_1 + \cdots + \tau_k = m$. Consider the boundary value problem

$$\begin{aligned} \mathcal{L}(x, D_x)u(x) &= f(x), \qquad && x \in G, \\ \mathcal{B}(x, D_x)u(x) &= g(x), \qquad && x \in \partial G, \end{aligned} \tag{2.1}$$

where $\mathcal{B}$ is an $m \times k$-matrix of differential operators. For a given $\mathcal{L}$ we find a class of boundary conditions such that for an element $\mathcal{B}$ of the class the self-adjoint Green formula

$$(\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} = (u, \mathcal{L}v)_G + (\mathcal{Q}u, \mathcal{B}v)_{\partial G} \tag{2.2}$$

holds with some $m \times k$-matrix $\mathcal{Q}$ of differential operators for all $u, v \in C_c^\infty(\overline{G})$. It is supposed that $\mathcal{B}$ in (2.1) is from the mentioned class and the problem is elliptic.

2.2 Boundary conditions and self-adjoint Green formula

If necessary changing the enumeration of the rows and columns in $\mathcal{L}_{ij}(x, D_x)$, we may always arrange that $\tau = \tau_1 \geq \tau_2 \geq \cdots \geq \tau_k$. In what follows we suppose that this has been done. Denote by K_s, $s = 1, \ldots, \tau$, the number of values j such that $\tau_j \geq \tau - s + 1$. Then $K_1 + \cdots + K_\tau = \tau_1 + \cdots + \tau_k = m$ and $K_1 \leq \cdots \leq K_\tau \leq k$. On the boundary ∂G we introduce the $m \times k$-matrix

$$\mathcal{D} = \begin{pmatrix} \mathbf{D}^1 \\ \vdots \\ \mathbf{D}^\tau \end{pmatrix}, \tag{2.3}$$

where the block $\mathbf{D}^s$ consists of the rows

$$(\delta_{1,h},\dots,\delta_{k,h})D_\nu^{\tau_h-\tau+s-1},\qquad h=1,\dots,K_s;$$

here ν is the unit outward normal to ∂G and $D_\nu=-i\partial/\partial\nu$. With $\mathcal{L}(x,D_x)$ we associate the sesquilinear form

$$a(u,v)=\sum_{i,j=1}^k\sum_{|\eta|\le\tau_h}\sum_{|\mu|\le\tau_j}\int_G a_{ij}^{\eta\mu}(x)D_x^\mu u_j(x)\overline{D_x^\eta v_i(x)}\,dx. \tag{2.4}$$

The Green formula

$$a(u,v)=(\mathcal{L}u,v)_G+(\mathcal{N}u,\mathcal{D}v)_{\partial G} \tag{2.5}$$

holds for $u,v\in C_c^\infty(\overline{G})$ with some $m\times k$-matrix $\mathcal{N}(x,D_x)=\|\mathcal{N}_{qj}(x,D_x)\|$ of differential operators, $\operatorname{ord}\mathcal{N}_{qj}+\operatorname{ord}\mathcal{D}_{qh}\le\tau_h+\tau_j-1$; by $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_{\partial G}$ we denote the inner products on $L_2(G)$ and $L_2(\partial G)$.

Remark 2.1. The proof of the Green formula (2.5) is standard. Using "local maps," we can consider G as the half-space $\mathbb{R}^n_-=\{x\in\mathbb{R}^n,x_n<0\}$ (cf. [5], [6]). Then $D_\nu=D_{x_n}$ and $\mathcal{L}_{ij}(x,D_x)=\sum_{|\alpha|+\beta\le\tau_i}\sum_{|\mu|\le\tau_j}D_\tau^\alpha D_\nu^\beta a_{ij}^{(\alpha\beta)\mu}(x)D_x^\mu$, where $D_\tau=D_{x_1}D_{x_2}\dots D_{x_{n-1}}$. Integrating by parts and changing the order of summation, we obtain

$$(\mathcal{L}u,v)_G=\sum_{i,j=1}^k\sum_{|\alpha|+\beta\le\tau_i}\sum_{|\mu|\le\tau_j}\left(D_\tau^\alpha D_\nu^\beta a_{ij}^{(\alpha\beta)\mu}(x)D_x^\mu u_j,v_i\right)_G=a(u,v)$$

$$+\sum_{s=1}^\tau\sum_{i=1}^{K_s}\sum_{j=1}^k\sum_{|\mu|\le\tau_j}\sum_{|\alpha|+\beta\le\tau_j,\beta\ge s}\left(D_\tau^\alpha D_\nu^{\beta-s+\tau-\tau_i}a_{ij}^{(\alpha\beta)\mu}(x)D_x^\mu u_j,D_\nu^{\tau_i-\tau+s-1}v_i\right)_{\partial G},$$

where $u,v\in C_c^\infty(\overline{G})$. This implies (2.5).

Definition 2.2. *A matrix $\mathcal{P}=\mathcal{P}(x,D_x)$ is called a Dirichlet system on the boundary ∂G if there exists an $m\times m$-matrix $\mathcal{R}=\mathcal{R}(x,D_x)$ satisfying the following conditions.*

(i) $\mathcal{P}(x,D_x)=\mathcal{R}(x,D_x)\mathcal{D}(x,D_x)$, where $\mathcal{D}$ is given in (2.3).

(ii) The matrix $\mathcal{R}$ consists of $K_p\times K_s$-blocks $\mathcal{R}_{[p,s]}$ $(p,s=1,\dots,\tau)$. The elements of $\mathcal{R}_{[p,s]}$, $s\le p$, are tangential differential operators with smooth coefficients on ∂G of order not higher than $p-s$, while the elements of $\mathcal{R}_{[p,s]}$, $s>p$, are zeros. The $\mathcal{R}_{[p,p]}(x)$ are nondegenerate matrices, $|\det\mathcal{R}_{[p,p]}(x)|>\varepsilon>0$ for $x\in\partial G$.

If in particular $k = 1$, then $\mathcal{P}(x, D_x)$ is the usual Dirichlet system of order τ (see [7],[8],[9]).

Remark 2.3. *The operator $\mathcal{R}$ from Definition 2.2 has an inverse $\mathcal{R}^{-1}$, which is a matrix of differential operators of the same structure as $\mathcal{R}$. The block $\mathcal{R}^{-1}_{[p,p]}(x)$ of $\mathcal{R}^{-1}(x, D_x)$ is the inverse matrix $(\mathcal{R}_{[p,p]}(x))^{-1}$, $p = 1, \ldots, \tau$. The blocks $\mathcal{R}^{-1}_{[p,s]}$, $s < p$, of sizes $K_p \times K_s$ can be successively found from the relations*

$$\mathcal{R}^{-1}_{[p,s]}(x, D_x) = \mathcal{R}^{-1}_{[p,p]}(x) \sum_{\ell=s}^{p-1} (-\mathcal{R}_{[p,\ell]}(x, D_x)) \mathcal{R}^{-1}_{[\ell,s]}(x, D_x).$$

The blocks $\mathcal{R}^{-1}_{[p,s]}$, $s > p$, consist of zeros.

Let

$$\mathcal{P}(x, D_x) = \mathcal{R}(x, D_x)\mathcal{D}(x, D_x) \tag{2.6}$$

be a Dirichlet system on ∂G. We set

$$\mathcal{T}(x, D_x) = \mathcal{R}^{-1}_*(x, D_x)\mathcal{N}(x, D_x), \tag{2.7}$$

where $\mathcal{R}^{-1}_*(x, D_x)$ is the formally adjoint differential operator to $\mathcal{R}^{-1}(x, D_x)$ and $\mathcal{N}(x, D_x)$ is from the Green formula (2.5). Introduce $m \times k$-matrices $\mathcal{B}$ and $\mathcal{Q}$ such that

$$(\mathcal{B}_{q1}, \ldots, \mathcal{B}_{qk}) = (\mathcal{T}_{q1}, \ldots, \mathcal{T}_{qk}), \quad (\mathcal{Q}_{q1}, \ldots, \mathcal{Q}_{qk}) = (\mathcal{P}_{q1}, \ldots, \mathcal{P}_{qk}) \tag{2.8}$$

for some numbers q, $1 \le q \le m$, while

$$(\mathcal{B}_{q1}, \ldots, \mathcal{B}_{qk}) = (\mathcal{P}_{q1}, \ldots, \mathcal{P}_{qk}), \quad (\mathcal{Q}_{q1}, \ldots, \mathcal{Q}_{qk}) = -(\mathcal{T}_{q1}, \ldots, \mathcal{T}_{qk}) \tag{2.9}$$

for the remaining rows of $\mathcal{B}$ and $\mathcal{Q}$. Therefore,

$$\begin{aligned}(\mathcal{N}u, \mathcal{D}v)_{\partial G} - (\mathcal{D}u, \mathcal{N}v)_{\partial G} &= (\mathcal{N}u, \mathcal{R}^{-1}\mathcal{P}v)_{\partial G} - (\mathcal{R}^{-1}\mathcal{P}u, \mathcal{N}v)_{\partial G} \\ &= (\mathcal{T}u, \mathcal{P}v)_{\partial G} - (\mathcal{P}u, \mathcal{T}v)_{\mathcal{G}} = (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (\mathcal{Q}u, \mathcal{B}v)_{\partial G}.\end{aligned} \tag{2.10}$$

Since $\mathcal{L}$ is formally self-adjoint, the form (2.4) is symmetric (i.e. $a(u, v) = \overline{a(v, u)}$). From (2.5) we obtain

$$(\mathcal{L}u, v)_G + (\mathcal{N}u, \mathcal{D}v)_{\partial G} = (u, \mathcal{L}v)_G + (\mathcal{D}u, \mathcal{N}v)_{\partial G}. \tag{2.11}$$

Together with (2.10) this leads to the Green formula (2.2).

2.3 Limit operators

Let r, $r = 1, \dots, N$, be a fixed number. We write the superscript r at $\mathcal{L}$, $\mathcal{R}$, and other operators if they are written in the coordinates (y,t) inside the semicylinder $\overline{\Pi}^r_+$. Let $\mathcal{L}^r = \|\mathcal{L}^r_{ij}\|$ and let

$$\mathcal{L}^r_{ij}(y,t,D_y,D_t) = \sum_{|\eta|+\mu \le \tau_j+\tau_h} \ell^{\eta\mu}_{ij}(y,t) D^\eta_y D^\mu_t. \tag{2.12}$$

We set $\psi_T(y,t) \equiv \psi(t-T)$ for $(y,t) \in \overline{\Pi}^r_+$, where $\psi \in C^\infty(\mathbb{R})$ is a cutoff function such that $\psi(t) = 0$ for $t < 1$ and $\psi(t) = 1$ for $t > 2$.

Definition 2.4. *We say that $\mathcal{L}$ is stabilizing in the semicylinder Π^r_+ if there exist functions $\mathbf{l}^{\eta\mu}_{hj}$ of $y \in \overline{\Omega}^r$ (limit coefficients) such that*

$$\lim_{T\to+\infty} \|\psi_T(\ell^{\eta\mu}_{ij} - \mathbf{l}^{\eta\mu}_{ij}); C^\infty(\overline{\Pi}^r_+)\| = 0, \quad |\eta|+\mu \le \tau_i+\tau_j, \quad i,j = 1,\dots,k. \tag{2.13}$$

Definition 2.5. *Let $\mathcal{F}(x,D_x) = \|\mathcal{F}_{qj}(x,D_x)\|$ be an operator given on the boundary ∂G. Write down $\mathcal{F}_{qh}$ in the local coordinates (y,t):*

$$\mathcal{F}^r_{qj}(y,t,D_y,D_t) = \sum_{|\alpha|+\beta \le \operatorname{ord}\mathcal{F}_{qj}} f^{\alpha\beta}_{qj}(y,t) D^\alpha_y D^\beta_t.$$

We say that $\mathcal{F}$ is stabilizing in Π^r_+ if there exist functions $\mathbf{f}^{\alpha\beta}_{qj}$ of $y \in \partial\Omega^r$ such that

$$\lim_{T\to+\infty} \|\psi_T(f^{\alpha\beta}_{qj} - \mathbf{f}^{\alpha\beta}_{qj}); C^\infty(\partial\Omega^r \times \mathbb{R}_+)\| = 0$$

for $|\alpha|+\beta \le \operatorname{ord}\mathcal{F}_{qj}$ and for all values of q and j; here $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$.

Since the coefficients of $\mathcal{D}^r_{qj}$ do not depend on t, the operator $\mathcal{D}(x,D_x)$ from (2.3) is stabilizing in $\Pi^1_+,\dots,\Pi^N_+$. Assume that $\mathcal{L}$ is stabilizing in Π^r_+. The stabilization of $\mathcal{N}(x,D_x)$ in Π^r_+ is guaranteed by (2.13) (the coefficients of $\mathcal{N}^r(y,t,D_y,D_t)$ are expressed in terms of $\ell^{\eta\mu}_{ij}$; see Remark 2.1). Therefore an operator $\mathcal{B}(x,D_x)$ constructed of the rows of $\mathcal{N}(x,D_x)$ and $\mathcal{D}(x,D_x)$ is stabilizing in Π^r as well; this case $\mathcal{R}(x,D_x) \equiv I$, see (2.6), (2.7) and (2.8), (2.9). In the general case we assume the stabilization of $\mathcal{R}(x,D_x) = \|\mathcal{R}_{hq}(x,D_x)\|^m_{h,q=1}$. Then the operator $\mathcal{R}^{-1}$ is stabilizing (see Remark 2.3), we get the stabilization of the operators $\mathcal{P}$, $\mathcal{T}$, $\mathcal{B}$, and $\mathcal{Q}$.

Let the elements $L^r_{ij}(y, D_y, D_t)$ of the limit operator $L^r = \|L^r_{ij}\|$ be given by the right-hand side of (2.12) with $\ell^{\eta\mu}_{ij}$ replaced by $\mathbf{l}^{\eta\mu}_{ij}$. Likewise, changing the coefficients to the limit ones, we define the limit operators N^r, R^r, B^r and etc. The relations $P^r = R^r\mathcal{D}^r$ and $T^r = (R^r)^{-1}_* N^r$ are fulfilled, where $(R^r)^{-1}_*$ is formally adjoint to $(R^r)^{-1}$. From (2.8) and (2.9) it follows that the matrix B^r and Q^r consist of the rows of P^r and T^r. The Green formula

$$(L^r u, v)_{\Pi^r} + (B^r u, Q^r v)_{\partial\Pi^r} = (u, L^r v)_{\Pi^r} + (Q^r u, B^r v)_{\partial\Pi^r} \tag{2.14}$$

is valid in the cylinder $\Pi^r = \Omega^r \times \mathbb{R}$, where $u, v \in C^\infty_c(\overline{\Pi}^r)$. We assume that the limit problem

$$\begin{aligned} L^r(y, D_y, D_t)u(y,t) &= F(y,t), \quad (y,t) \in \Pi^r, \\ B^r(y, D_y, D_t)u(y,t) &= G(y,t), \quad (y,t) \in \partial\Pi^r, \end{aligned} \tag{2.15}$$

is elliptic.

Denote by $W^l_\gamma(\Pi^r)$ the space with norm $\|e_\gamma \cdot; H^l(\Pi^r)\|$, where $H^l(\Pi^r)$ is the Sobolev space, $e_\gamma : (y,t) \mapsto \exp(\gamma t)$, and $\gamma \in \mathbb{R}$. For $l \geq \tau$ we set

$$\begin{aligned} \mathcal{D}^l_\gamma(\Pi^r) &= \prod_{j=1}^k H^{l+\tau_j}_\gamma(\Pi^r), \\ \mathcal{R}^l_\gamma(\Pi^r) &= \prod_{i=1}^k H^{l-\tau_i}_\gamma(\Pi^r) \times \prod_{q=1}^m H^{l-\tau-\sigma_q-1/2}_\gamma(\partial\Pi^r) \end{aligned} \tag{2.16}$$

with $\sigma_q = \operatorname{ord} B^r_{qj} - \tau - \tau_j$, $\sigma_q < 0$. The map

$$A^r_\gamma = \{L^r, B^r\} : \mathcal{D}^\ell_\gamma(\Pi^r) \to \mathcal{R}^\ell_\beta(\Pi^r) \tag{2.17}$$

is continuous. We introduce the operator pencil

$$\mathbb{C} \ni \lambda \mapsto \mathfrak{A}^r(\lambda) = \{L^r(y, D_y, \lambda), B^r(y, D_y, \lambda)\} \tag{2.18}$$

in the domain Ω^r. The spectrum of $\mathfrak{A}^r$ is symmetric about the real line and consists of normal eigenvalues. Any strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq h < \infty\}$ contains at most finitely many points of the spectrum. Denote by $\lambda_{-\nu^0}, \dots, \lambda_{\nu^0}$ with $\nu^0 \geq 0$ all the real eigenvalues of $\mathfrak{A}^r$ (if the number of real eigenvalues is even, then λ_0 is absent). We enumerate the nonreal eigenvalues so that $0 < \operatorname{Im}\lambda_{\nu^0+1} \leq \operatorname{Im}\lambda_{\nu^0+2} \leq \dots$ and $\lambda_\nu = \overline{\lambda}_{-\nu}$, where $\nu = \nu^0+1, \nu^0+2, \dots$.

Let $\{\varphi_\nu^{(0,j)},\dots,\varphi_\nu^{(\varkappa_{j\nu}-1,j)}; j=1,\dots,J_\nu := \dim\ker\mathfrak{A}^r(\lambda_\nu)\}$ be a canonical system of Jordan chains of the pencil $\mathfrak{A}^r$ corresponding to λ_ν, i.e. $\varphi_\nu^{(0,j)}$ is an eigenvector and $\varphi_\nu^{(1,j)},\dots,\varphi_\nu^{(\varkappa_{j\nu}-1,j)}$ are associated vectors (e.g., see [10]). The functions

$$u_\nu^{(\sigma,j)}(y,t)=\exp(i\lambda_\nu t)\sum_{\ell=0}^{\sigma}\frac{1}{\ell!}(it)^\ell\varphi_\nu^{(\sigma-\ell,j)}(y) \tag{2.19}$$

with $\sigma=0,\dots,\varkappa_{j\nu}-1$ satisfy the homogeneous problem (2.15). We introduce the form

$$q^r(u,v):=(L^r u,v)_{\Pi^r}+(B^r u,Q^r v)_{\partial\Pi^r}-(u,L^r v)_{\Pi^r}-(Q^r u,B^r v)_{\partial\Pi^r}. \tag{2.20}$$

It is obvious that $q^r(u,v)=-\overline{q^r(v,u)}$ and $q^r(u,u)\in i\mathbb{R}$. The Green formula (2.14) extends by continuity to the functions $U\in\mathcal{D}^l_\gamma(\Pi^r)$ and $V\in\mathcal{D}^l_{-\gamma}(\Pi^r)$, therefore $q^r(U,V)=0$.

Proposition 2.6 (see [3]). (i) *Let $\chi\in C^\infty(\mathbb{R})$, $\chi(t)=1$ for $t\geq 2$ and $\chi(t)=0$ for $t\leq 1$. One can choose Jordan chains $\{\varphi_\nu^{(\sigma,j)}\}$ to satisfy the following conditions:*

$$q^r(\chi u_\nu^{(\sigma,j)},\chi u_\mu^{(\tau,p)})=i\delta_{-\nu,\mu}\delta_{j,p}\delta_{\varkappa_{j\nu}-1-\sigma,\tau},\quad |\nu|>\nu_0,|\mu|>\nu_0, \tag{2.21}$$

$$q^r(\chi u_\nu^{(\sigma,j)},\chi u_\mu^{(\tau,p)})=\pm i\delta_{\nu,\mu}\delta_{j,p}\delta_{\varkappa_{j\nu}-1-\sigma,\tau},\quad |\nu|\leq\nu_0,|\mu|\leq\nu_0, \tag{2.22}$$

$$q^r(\chi u_\nu^{(\sigma,j)},\chi u_\mu^{(\tau,p)})=0,\quad |\nu|\leq\nu_0,|\mu|>\nu_0, \tag{2.23}$$

where the functions $u_\nu^{(\sigma,j)}$ are given in (2.19). In (2.22) the sign depends on ν and j (and cannot be taken arbitrarily). The equality (2.23) remains true for arbitrary choice of Jordan chains and for any superscripts. The conditions (2.21) – (2.23) do not depend on the choice of χ.
(ii) *The map (2.17) is an isomorphism if and only if the line $\mathbb{R}+i\gamma=\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda=\gamma\}$ is free of the spectrum of the pencil $\mathfrak{A}^r$.*

Let $\gamma\geq 0$. Denote by $\mathcal{W}_\gamma(\Pi^r)$ the linear span of the functions $\{u_\nu^{(\sigma,j)}: |\operatorname{Im}\lambda|\leq\gamma\}$. It is known (see [3]) that the total algebraic multiplicity of all the eigenvalues of the pencil $\mathfrak{A}^r$ in the strip $\{\lambda\in\mathbb{C}:|\operatorname{Im}\lambda|\leq\gamma\}$ is even for any $\gamma\geq 0$; we denote the multiplicity by $2M^r(\equiv 2M^r_\gamma)$. There is a basis

$$u_1^+,\dots,u_{M^r}^+,u_1^-,\dots,u_{M^r}^- \tag{2.24}$$

in the space $\mathcal{W}_\gamma(\Pi^r)$ obeying

$$q^r(\chi u_j^\pm, \chi u_h^\pm) = \mp i\delta_{j,h}, \quad q^r(\chi u_j^\pm, \chi u_h^\mp) = 0, \quad j, h = 1, \dots, M^r, \tag{2.25}$$

(see [3],[11]); here χ is the cut-off function from Proposition 2.6. One can consider the cylinder Π^r as a generalized waveguide. The space $\mathcal{W}_\gamma(\Pi^r)$ is called the space of waves. The quantity $iq^r(\chi u, \chi u)$ represents the energy flow transferred by the wave $u \in \mathcal{W}_\gamma(\Pi^r)$ through the cross-section Ω^r of the cylinder Π^r. Thus $u_1^+, \dots, u_{M^r}^+$ are incoming waves and $u_1^-, \dots, u_{M^r}^-$ are outgoing waves for the problem (2.15).

The following proposition is a variant of Proposition 3.1.4 and Theorem 3.2.1 from [3].

Proposition 2.7. *Assume that $\gamma > 0$, the line $\mathbb{R}+i\gamma$ is free of the spectrum of the pencil $\mathfrak{A}^r$, and $\{F, G\} \in \mathcal{R}^l_\gamma(\Pi^r) \cap \mathcal{R}^l_{-\gamma}(\Pi^r)$. Then a solution $u \in \mathcal{D}^\ell_{-\gamma}(\Pi^r)$ to the problem (2.15) admits the representation*

$$u = \sum_{j=1}^{M^r} \{a_j(F, G)u_j^+ + b_j(F, G)u_j^-\} + v, \tag{2.26}$$

where the functions $u_j^\pm$ form a basis in $\mathcal{W}_\gamma(\Pi^r)$ and satisfy (2.25), v is a solution to the problem (2.15) in $\mathcal{D}^l_\gamma(\Pi^r)$. The functionals $a_1, \dots, a_{M^r}$ and $b_1, \dots, b_{M^r}$ are continuous on $\mathcal{R}^l_\gamma(\Pi^r) \cap \mathcal{R}^l_{-\gamma}(\Pi^r)$ and

$$\begin{aligned} a_j(F, G) &= i(F, u_j^+)_{\Pi^r} + i(G, Q^r u_j^+)_{\partial\Pi^r}, \\ b_j(F, G) &= -i(F, u_j^-)_{\Pi^r} - i(G, Q^r u_j^-)_{\partial\Pi^r}, \end{aligned} \tag{2.27}$$

where Q^r is the same as in the Green formula (2.5).

3 The structure of solutions to the problem (2.1)

3.1 Construction of a model problem in Π^r

In this subsection we construct a differential operator $\{\mathfrak{L}_T^r, \mathfrak{B}_T^r\}$ in Π^r such that the following conditions are satisfied: (i) $\{\mathfrak{L}_T^r, \mathfrak{B}_T^r\}$ coincides with $\{\mathcal{L}, \mathcal{B}\}$ on the set $\{(y,t) \in \overline{\Pi}^r : t > T + 3\}$; (ii) $\{\mathfrak{L}_T^r, \mathfrak{B}_T^r\}$ coincides with $\{L^r, B^r\}$

on the set $\{(y,t) \in \overline{\Pi}^r : t < T\}$; (iii) the norm $\|\Delta^r_T; \mathcal{D}^l_\gamma(\Pi^r) \to \mathcal{R}^l_\gamma(\Pi^r)\|$ of the operator

$$\Delta^r_T = \{L^r, B^r\} - \{\mathfrak{L}^r_T, \mathfrak{B}^r_T\} \tag{3.1}$$

tends to zero as $T \to +\infty$; (iv) for $u, v \in C^\infty_c(\overline{\Pi}^r)$ and sufficiently large T the self-adjoint Green Formula

$$(\mathfrak{L}^r_T u, v)_{\Pi^r} + (\mathfrak{B}^r_T u, \mathfrak{Q}^r_T v)_{\partial\Pi^r} = (u, \mathfrak{L}^r_T v)_{\Pi^r} + (\mathfrak{Q}^r_T u, \mathfrak{B}^r_T v)_{\partial\Pi^r} \tag{3.2}$$

holds with some $m \times k$-matrix $\mathfrak{Q}^r_T$ of differential operators.

Recall that $\psi_T(y,t)$ is a cutoff function, $\psi_T(y,t) \equiv \psi(t-T)$ for $(y,t) \in \overline{\Pi}^r_+$, where $\psi \in C^\infty(\mathbb{R})$, $\psi(t) = 0$ for $t < 1$ and $\psi(t) = 1$ for $t > 2$. Let $\mathfrak{L}^r_T = L^r - \psi_T(L^r - \mathcal{L})\psi_T$, where the operator $\psi_T \mathcal{L} \psi_T$ is extended from $\overline{\Pi}^r_+$ to the whole cylinder $\overline{\Pi}^r$ by zero. First we find a Dirichlet system $\mathcal{P}^r_T$ and an operator $\mathcal{T}^r_T$ such that

$$(\mathfrak{L}^r_T u, v)_{\Pi^r} + (\mathcal{P}^r_T u, \mathcal{T}^r_T v)_{\partial\Pi^r} = (u, \mathfrak{L}^r_T v)_{\Pi^r} + (\mathcal{T}^r_T u, \mathcal{P}^r_T v)_{\partial\Pi^r} \tag{3.3}$$

for $u, v \in C^\infty_c(\overline{\Pi}^r)$. Then we compose $\mathfrak{B}^r_T$ and $\mathfrak{Q}^r_T$ from the rows of $\mathcal{P}^r_T$ and $\mathcal{T}^r_T$ (by analogy with (2.8), (2.9)) and derive (3.2) from (3.3).

Denote $\mathcal{N}^r_T = N^r - \psi_T(N^r - \mathcal{N})\psi_T$ and $\mathcal{D}^r_T = \mathcal{D}^r - \psi_T(\mathcal{D}^r - \mathcal{D})\Psi_T$ (the operators $\psi_T \mathcal{N} \psi_T$ and $\psi_T \mathcal{D} \psi_T$ are extended to $\overline{\Pi}^r$ by zero). It is clear that $\mathcal{D}^r_T \equiv \mathcal{D}^r$ and $\mathcal{D}^r_T$ is the Dirichlet system on $\partial\Pi^r$. Since $\mathcal{D}^r$ consists of normal derivatives, we have $[\mathcal{D}^r, \psi_T] = 0$ on $\partial\Pi^r$; here $[a,b] = ab - ba$. Thus, substituting $\psi_T u$ and $\psi_T v$ for u and v in (2.11), we obtain

$$\begin{aligned}(\psi_T \mathcal{L} \psi_T u, v)_{\Pi^r} &+ (\psi_T \mathcal{N} \psi_T u, \mathcal{D}^r v)_{\partial\Pi^r} \\ &= (u, \psi_T \mathcal{L} \psi_T v)_{\Pi^r} + (\mathcal{D}^r u, \psi_T \mathcal{N} \psi_T v)_{\partial\Pi^r}.\end{aligned} \tag{3.4}$$

By the same arguments from

$$(L^r u, v)_{\Pi^r} + (N^r u, \mathcal{D}^r v)_{\partial\Pi^r} = (u, L^r v)_{\Pi^r} + (\mathcal{D}^r u, N^r v)_{\partial\Pi^r} \tag{3.5}$$

we get

$$\begin{aligned}(\psi_T L^r \psi_T u, v)_{\Pi^r} &+ (\psi_T N^r \psi_T u, \mathcal{D}^r v)_{\partial\Pi^r} \\ &= (u, \psi_T L^r \psi_T v)_{\Pi^r} + (\mathcal{D}^r u, \psi_T N^r \psi_T v)_{\partial\Pi^r}.\end{aligned} \tag{3.6}$$

Adding (3.4) and (3.5) and subtracting (3.6), we arrive at the formula

$$(\mathfrak{L}^r_T u, v)_{\Pi^r} + (\mathcal{N}^r_T u, \mathcal{D}^r_T v)_{\partial\Pi^r} = (u, \mathfrak{L}^r v)_{\Pi^r} + (\mathcal{D}^r_T u, \mathcal{N}^r_T v)_{\partial\Pi^r}. \tag{3.7}$$

Recall that $\mathcal{P} = \mathcal{R}\mathcal{D}$ and $P^r = R^r\mathcal{D}^r$. For sufficiently large T we put $\mathcal{R}^r_T = R^r + \psi_T(\mathcal{R} - R^r)\psi_T$. Due to the stabilization of $\mathcal{R}$ in Π^r_+, the matrix $\mathcal{P}^r_T \equiv \mathcal{R}^r_T\mathcal{D}^r$ is a Dirichlet system on $\partial\Pi^r$, and there exists a differential operator $(\mathcal{R}^r_T)^{-1}$ such that $(\mathcal{R}^r_T)^{-1}\mathcal{R}^r_T = \mathcal{R}^r_T(\mathcal{R}^r_T)^{-1} = I$; see Remark 2.3. Let $\mathcal{T}^r_T = (\mathcal{R}^r_T)^{-1}_*\mathcal{N}^r_T$. We have

$$
\begin{aligned}
&(\mathcal{N}^r_T u, \mathcal{D}^r_T v)_{\partial\Pi^r} - (\mathcal{D}^r_T u, \mathcal{N}^r_T v)_{\partial\Pi^r} \\
&\quad = ((\mathcal{R}^r_T)^{-1}_*\mathcal{N}^r_T u, \mathcal{R}^r_T\mathcal{D}^r_T v)_{\partial\Pi^r} - (\mathcal{R}^r_T\mathcal{D}^r_T u, (\mathcal{R}^r_T)^{-1}_*\mathcal{N}^r_T v)_{\partial\Pi^r} \\
&\qquad\qquad = (\mathcal{T}^r_T u, \mathcal{P}^r_T v)_{\partial\Pi^r} - (\mathcal{P}^r_T u, \mathcal{T}^r_T v)_{\partial\Pi^r}.
\end{aligned}
$$

Together with (3.7) this implies (3.3). Composing the matrices $\mathfrak{B}^r_T$ and $\mathfrak{Q}^r_T$ from the rows of $\mathcal{P}^r_T$ and $\mathcal{T}^r_T$ by the same rule as in (2.8) and (2.9), we obtain the Green formula (3.2).

By the construction of $\{\mathfrak{L}^r_T, \mathfrak{B}^r_T\}$ the conditions (i), (ii), and (iv) given in the beginning of this subsection are satisfied. Due to the stabilization of $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ in Π^r_+ we have

$$\lim_{T\to+\infty} \|\Delta^r_T; \mathcal{D}^l_\gamma(\Pi^r) \to \mathcal{R}^l_\gamma(\Pi^r)\| = 0 \tag{3.8}$$

for all $\gamma \in \mathbb{R}$, condition (iii) is fulfilled.

In the cylinder Π^r we consider the model problem

$$
\begin{aligned}
\mathfrak{L}^r_T(y, t, D_y, D_t)u(y, t) &= \mathfrak{F}(y, t), \quad (y, t) \in \Pi^r, \\
\mathfrak{B}^r_T(y, t, D_y, D_t)u(y, t) &= \mathfrak{G}(y, t), \quad (y, t) \in \partial\Pi^r.
\end{aligned} \tag{3.9}
$$

3.2 The structure of solutions to the model problem (3.9)

Taking into account (3.8) and the invertibility of the limit operator (2.17) (see Proposition 2.6), we get the following assertion.

Proposition 3.1. *Let the operators $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ stabilize in Π^r_+ and let the line $\mathbb{R} + i\gamma$ contain no eigenvalues of the pencil $\mathfrak{A}^r$. Assume that T is sufficiently large. Then the operator*

$$\{\mathfrak{L}^r_T, \mathfrak{B}^r_T\} : \mathcal{D}^\ell_\gamma(\Pi^r) \to \mathcal{R}^\ell_\gamma(\Pi^r)$$

of the problem (3.9) implements an isomorphism.

We now introduce functions $z_j^\pm$, which play the same role for the problem (3.9) as the waves $u_j^\pm$ play for the limit problem (2.15). Suppose that the assumptions of Proposition 3.1 are fulfilled. We set

$$z_j^\pm = u_j^\pm + \sum_{q=1}^{\infty} ((A_{-\gamma}^r)^{-1}\Delta_T^r)^q u_j^\pm, \quad j = 1, \dots, M^r, \tag{3.10}$$

where the waves $\{u_j^\pm : j = 1, \dots, M^r\}$ form a basis in $\mathcal{W}_\gamma(\Pi^r)$ obeying (2.25). (Recall that $2M^r (\equiv 2M^r(\gamma))$ is the total algebraic multiplicity of all the eigenvalues of the pencil $\mathfrak{A}^r$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \le \gamma\}$.)

Let us discuss the equality (3.10). Note that $\psi_{T-2} u_j^\pm \in \mathcal{D}_{-\gamma}^l(\Pi^r)$ and $\Delta_T^r u_j^\pm = \Delta_T^r \psi_{T-2} u_j^\pm$ with the same cutoff function ψ_T as in the previous section. By virtue of (3.8) the norm of operator $(A_{-\gamma}^r)^{-1}\Delta_T^r : \mathcal{D}_{-\gamma}^l(\Pi^r) \to \mathcal{D}_{-\gamma}^l(\Pi^r)$ is small; $(A_{-\gamma}^r)^{-1}$ is bounded because the spectrum of $\mathfrak{A}^r$ is symmetric about the real axis, see Proposition 2.6, (ii). The series $\sum_{q=1}^\infty ((A_{-\gamma}^r)^{-1}\Delta_T^r)^q u_j^\pm$ converges in the norm of $\mathcal{D}_{-\gamma}^l(\Pi^r)$. Consequently,

$$z_j^\pm = u_j^\pm \mod \mathcal{D}_{-\gamma}^l(\Pi^r). \tag{3.11}$$

Proposition 3.2. *Let the assumptions of Proposition 3.1 be fulfilled. Then the functions*

$$z_1^+, \dots, z_{M^r}^+, z_1^-, \dots, z_{M^r}^- \tag{3.12}$$

defined by (3.10) are linearly independent modulo $\mathcal{D}_{-\gamma}^l(\Pi^r)$ solutions to the homogeneous problem (3.9).

Proof. It is easy to see that the functions $u_\nu^{(\sigma,j)}$ forming the linear span $\mathcal{W}_\gamma(\Pi^r)$ are linearly independent modulo $\mathcal{D}_{-\gamma}^l(\Pi^r)$; see (2.19). Thus the elements of the basis $u_1^+, \dots, u_{M^r}^+, u_1^-, \dots, u_{M^r}^-$ in $\mathcal{W}_\gamma(\Pi^r)$ are linearly independent modulo $\mathcal{D}_{-\gamma}^l(\Pi^r)$. Together with the relations (3.11) this implies the linear independence of $z_j^\pm$, $j = 1, \dots, M^r$, modulo $\mathcal{D}_{-\gamma}^l(\Pi^r)$.

Let us show that $z_j^\pm$ satisfy the homogeneous problem (3.9). Consider the equation

$$\{\mathfrak{L}_T^r, \mathfrak{B}_T^r\} w = \Delta_T^r u_j^\pm. \tag{3.13}$$

The inclusion $\Delta_T^r u_j^\pm \in \mathcal{R}_{-\gamma}^l(\Pi^r)$ holds for $j = 1, \dots, M^r$. The line $\mathbb{R} - i\gamma$ is free of the spectrum of $\mathfrak{A}^r$ (because the spectrum of $\mathfrak{A}^r$ is symmetric about the real line). By Proposition 3.1 there exists a unique solution $w \in \mathcal{D}_{-\gamma}^l(\Pi^r)$ to

the problem (3.13). Multiplying (3.13) from left by $(A^r_{-\gamma})^{-1}$ and using (3.1), we get

$$(I-(A^r_{-\gamma})^{-1}\Delta^r_T)w=(A^r_{-\gamma})^{-1}\Delta^r_T u^{\pm}_j.$$

Thanks to (3.8) the operator $I-(A^r_{-\gamma})^{-1}\Delta^r_T$ is invertible. We write $(I-(A^r_{-\gamma})^{-1}\Delta^r_T)^{-1}$ as the Neumann series and obtain

$$w=\big(I-(A^r_{-\gamma})^{-1}\Delta^r_T\big)^{-1}(A^r_{-\gamma})^{-1}\Delta^r_T u^{\pm}_j=\sum_{q=1}^{\infty}((A^r_{-\gamma})^{-1}\Delta^r_T)^q u^{\pm}_j.$$

Keeping in mind that $\{L^r,B^r\}u^{\pm}_j=0$ and (3.1), we deduce from (3.13) that $\{\mathfrak{L}^r_T,\mathfrak{B}^r_T\}(w+u^{\pm}_j)=0$. It remains to note that $z^{\pm}_j\equiv w+u^{\pm}_j$. □

Introduce the form

$$p^r_T(u,v)=(\mathfrak{L}^r_T u,v)_{\Pi^r}+(\mathfrak{B}^r_T u,\mathfrak{Q}^r_T v)_{\partial\Pi^r} \\ -(u,\mathfrak{L}^r_T v)_{\Pi^r}-(\mathfrak{Q}^r_T u,\mathfrak{B}^r_T v)_{\partial\Pi^r}. \tag{3.14}$$

It is easy to see that $p^r_T(u,v)=0$ for $u\in\mathcal{D}^l_{-\gamma}(\Pi^r)$ and $v\in\mathcal{D}^l_{\gamma}(\Pi^r)$ (indeed, for such functions the Green formula (3.2) is fulfilled).

Proposition 3.3. *Let the assumptions of Proposition 3.1 be fulfilled. Then the functions (3.12) satisfy the conditions*

$$p^r_T(\chi z^{\pm}_h,\chi z^{\pm}_j)=\mp i\delta_{h,j},\quad p^r_T(\chi z^{\pm}_h,\chi z^{\mp}_j)=0,\quad h,j=1,\dots,M^r, \tag{3.15}$$

where $\chi\in C^\infty(\mathbb{R})$, $\chi(t)=1$ for $t\geq 2$ and $\chi(t)=0$ for $t\leq 1$.The equalities (3.15) do not depend on the choice of χ.

Proof. Since the waves $u^{\pm}_j$ satisfy the homogeneous problem (2.15), we have $q^r(u^{\pm}_h,u^{\pm}_j)=0$ and

$$-q^r(\chi u^{\pm}_h,u^{\pm}_j)=q^r((1-\chi)u^{\pm}_h,u^{\pm}_j), \tag{3.16}$$

where q^r is from (2.20). Note that operator $\{L^r,B^r\}$ coincide with $\{\mathfrak{L}^r_T,\mathfrak{B}^r_T\}$ on the support of $(1-\chi)u^{\pm}_h$. This allows us to write (3.16) in the form

$$-q^r(\chi u^{\pm}_h,u^{\pm}_j)=p^r_T((1-\chi)u^{\pm}_h,u^{\pm}_j). \tag{3.17}$$

Due to (3.11) and $\chi u^{\pm}_h\in\mathcal{D}^l_{\gamma}(\Pi^r)$ the Green formula (3.2) is valid on the pairs

$$\{u,v\}=\{(1-\chi)(z^{\pm}_h-u^{\pm}_h),z^{\pm}_j\},$$
$$\{u,v\}=\{(1-\chi)(z^{\pm}_h-u^{\pm}_h),z^{\pm}_j-u^{\pm}_j\},$$
$$\{u,v\}=\{(1-\chi)z^{\pm}_h,z^{\pm}_j-u^{\pm}_j\}.$$

Thus $p_T^r(u,v)=0$ on the same pairs, and

$$p_T^r((1-\chi)u_h^{\pm},u_j^{\pm})=p_T^r((1-\chi)z_h^{\pm},z_j^{\pm}). \tag{3.18}$$

Thanks to Proposition 3.2 we have $p_T^r(z_h^{\pm},z_j^{\pm})=0$. Therefore, from (3.17) and (3.18) we get

$$-q^r(\chi u_h^{\pm},u_j^{\pm})=-p_T^r(\chi z_h^{\pm},z_j^{\pm}).$$

Finally we obtain

$$q^r(\chi u_h^{\pm},\chi u_j^{\pm})=q^r(\chi u_h^{\pm},u_j^{\pm})=p_T^r(\chi z_h^{\pm},z_j^{\pm})=p_T^r(\chi z_h^{\pm},\chi z_j^{\pm}).$$

To establish the first equality in (3.15) it remains to use (2.25). In a similar way one can prove the second equality in (3.15). □

The first assertion of the following theorem is a variant of Theorem 6.2 from [1]; see also [2, Theorem 8.5.7].

Theorem 3.4. *Assume that the operators $\mathcal{L}$ and $\mathcal{R}$ stabilize in Π^r. Let $\gamma>0$ and let the line $\mathbb{R}+i\gamma$ be free of the spectrum of the pencil $\mathfrak{A}^r$. Then for sufficiently large T the following assertions hold.*
(i) *A solution $u\in\mathcal{D}^l_{-\gamma}(\Pi^r)$ to the problem (3.9) with right-hand side $\{\mathfrak{F},\mathfrak{G}\}\in\mathcal{R}^l_{\gamma}(\Pi^r)\cap\mathcal{R}^l_{-\gamma}(\Pi^r)$ admits the representation*

$$u=\sum_{j=1}^{M^r}\{a_j(\mathfrak{F},\mathfrak{G})\,z_j^{+}+b_j(\mathfrak{F},\mathfrak{G})\,z_j^{-}\}+v, \tag{3.19}$$

where v is a solution to the same problem in $\mathcal{D}^l_{\gamma}(\Pi^r)$, the waves $z_j^{\pm}$ are defined by (3.10), and $2M^r$ is the total algebraic multiplicity of the eigenvalues of the pencil $\mathfrak{A}^r$ in the strip $\{\lambda\in\mathbb{C}:|\operatorname{Im}|<\gamma\}$.
(ii) *The functionals $a_1,\dots,a_{M^r}$ and $b_1,\dots,b_{M^r}$ are continuous on $\mathcal{R}^l_{\gamma}(\Pi^r)\cap\mathcal{R}^l_{-\gamma}(\Pi^r)$ and*

$$\begin{aligned}a_j(\mathfrak{F},\mathfrak{G})&=i(\mathfrak{F},z_j^{+})_{\Pi^r}+i(\mathfrak{G},\mathfrak{Q}_T^r z_j^{+})_{\partial\Pi^r},\\ b_j(\mathfrak{F},\mathfrak{G})&=-i(\mathfrak{F},z_j^{-})_{\Pi^r}-i(\mathfrak{G},\mathfrak{Q}_T^r z_j^{-})_{\partial\Pi^r},\end{aligned} \tag{3.20}$$

where $\mathfrak{Q}_T^r$ is the same as in the Green formula (3.2).

Proof. (i) Since the spectrum of $\mathfrak{A}^r$ is symmetric about the real line, the conditions of theorem guaranty that the line $\mathbb{R}-i\gamma$ is free of the spectrum. From the second assertion of Proposition 2.6 and (3.8) it follows that

$$\begin{aligned}&\|(A^r_{\gamma})^{-1}\Delta_T^r;\mathcal{D}^{\ell}_{\gamma}(\Pi^r)\to\mathcal{D}^{\ell}_{\gamma}(\Pi^r)\|<1,\\ &\|(A^r_{-\gamma})^{-1}\Delta_T^r;\mathcal{D}^{\ell}_{-\gamma}(\Pi^r)\to\mathcal{D}^{\ell}_{-\gamma}(\Pi^r)\|<1.\end{aligned}$$

Solutions $u \in \mathcal{D}^\ell_{-\gamma}(\Pi^r)$ and $v \in \mathcal{D}^\ell_{\gamma}(\Pi^r)$ to the problem (3.9) satisfy the equations

$$u = (A^r_{-\gamma})^{-1}\Delta^r_T u + (A^r_{-\gamma})^{-1}\{\mathfrak{F},\mathfrak{G}\}, \quad v = (A^r_{\gamma})^{-1}\Delta^r_T v + (A^r_{\gamma})^{-1}\{\mathfrak{F},\mathfrak{G}\},$$

where Δ^r_T is from (3.1). Let us solve this equations by the method of successive approximations. We set

$$\begin{aligned} u_{n+1} &= (A^r_{-\gamma})^{-1}\Delta^r_T u_n + u_0, \quad u_0 = (A^r_{-\gamma})^{-1}\{\mathfrak{F},\mathfrak{G}\},\\ v_{n+1} &= (A^r_{\gamma})^{-1}\Delta^r_T v_n + v_0, \quad v_0 = (A^r_{\gamma})^{-1}\{\mathfrak{F},\mathfrak{G}\}. \end{aligned}$$

By Proposition 2.7 we have

$$u_0 = i\sum_{j=1}^{M^r}\{a_j(\mathfrak{F},\mathfrak{G})u^+_j + b_j(\mathfrak{F},\mathfrak{G})u^-_j\} + v_0. \tag{3.21}$$

Let us write the formulas (2.27) for $a_j(\mathfrak{F},\mathfrak{G})$ and $b_j(\mathfrak{F},\mathfrak{G})$ in the form

$$a_j(\mathfrak{F},\mathfrak{G}) = iq^r(v_0,u^+_j), \quad b_j(\mathfrak{F},\mathfrak{G}) = -iq^r(v_0,u^-_j),$$

where q^r is from (2.20). We prove by induction that

$$\begin{aligned} u_n = v_n + i\sum_{j=1}^{M^r}\sum_{m=0}^{n}((A^r_{-\gamma})^{-1}\Delta^r_T)^m\{q^r(v_{n-m},u^+_j)u^+_j \\ - q^r(v_{n-m},u^-_j)u^-_j\}. \end{aligned} \tag{3.22}$$

If $n = 0$ then (3.22) coincides with (3.21). We suppose that (3.22) holds for n and show that it remains valid for $n+1$. From (3.22) we get

$$\begin{aligned} &(A^r_{-\gamma})^{-1}\Delta^r_T u_n = (A^r_{-\gamma})^{-1}\Delta^r_T v_n \\ &+ i\sum_{j=1}^{M^r}\sum_{m=0}^{n}((A^r_{-\gamma})^{-1}\Delta^r_T)^{m+1}\{q^r(v_{n-m},u^+_j)u^+_j - q^r(v_{n-m},u^-_j)u^-_j\}. \end{aligned} \tag{3.23}$$

Using Proposition 2.7, we represent $(A^r_{-\gamma})^{-1}\Delta^r_T v_n$ in the form

$$\begin{aligned} &(A^r_{-\gamma})^{-1}\Delta^r_T v_n = (A^r_{\gamma})^{-1}\Delta^r_T v_n \\ &+ i\sum_{j=1}^{M^r}\big\{q^r\big((A^r_{-\gamma})^{-1}\Delta^r_T v_n,u^+_j\big)u^+_j - q^r\big((A^r_{\gamma})^{-1}\Delta^r_T v_n,u^-_j\big)u^-_j\big\}. \end{aligned} \tag{3.24}$$

Taking into account (3.21), (3.24) and the formulas for u_{n+1} and v_{n+1}, we pass from (3.23) to the equality (3.22) with n replaced by $n+1$. The formula (3.22) is proved.

Substituting $v_{n-m} = \sum_{h=0}^{n-m}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0$ into (3.22) we obtain

$$\begin{aligned} u_n = v_n + i\sum_{j=1}^{M^r}\sum_{m=0}^{n}((A_{-\gamma}^r)^{-1}\Delta_T^r)^m\Big\{q^r\big(\sum_{h=0}^{n-m}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0, u_j^+\big)u_j^+ \\ - q^r\big(\sum_{h=0}^{n-m}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0, u_j^-\big)u_j^-\Big\}. \end{aligned} \tag{3.25}$$

The series $\sum_{h=0}^{\infty}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0$ converges in the norm of $\mathcal{D}_\gamma^\ell(\Pi^r)$, moreover, $\{L^r, B^r\}\sum_{h=0}^{\infty}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0 \in \mathcal{R}_\gamma^\ell(\Pi^r)\cap\mathcal{R}_{-\gamma}^\ell(\Pi^r)$. Using the argument given after (3.10), we justify the passage to the limit in (3.25) as $n\to\infty$. As a result we get the representation (3.19), where the functions $u \in \mathcal{D}_{-\gamma}^\ell(\Pi^r)$ and $v \in \mathcal{D}_\gamma^\ell(\Pi^r)$ satisfy the problem (3.9), and

$$\begin{aligned} a_j(\mathfrak{F},\mathfrak{G}) &= iq^r\big(\sum_{h=0}^{\infty}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0, u_j^+\big), \\ b_j(\mathfrak{F},\mathfrak{G}) &= -iq^r\big(\sum_{h=0}^{\infty}((A_\gamma^r)^{-1}\Delta_T^r)^h v_0, u_j^-\big). \end{aligned}$$

The assertion (i) is proved.

Let us establish the formulas (3.20). Due to Proposition 3.2 we have

$$(\mathfrak{F}, z_j^\pm)_{\Pi^r} + (\mathfrak{G}, \mathfrak{Q}_T^r z_j^\pm)_{\partial\Pi^r} = p_T^r(u, z_j^\pm), \tag{3.26}$$

where u is the same as in (3.19). Let $\chi\in C^\infty(\mathbb{R})$, $\chi(t)=1$ for $t\geq 2$ and $\chi(t)=0$ for $t\leq 1$. From (3.11) and $(1-\chi)u_j^\pm \in D_\gamma^l(\Pi^r)$ we obtain $(1-\chi)z_j^\pm \in D_\gamma^l(\Pi^r)$. Then the inclusion $u\in\mathcal{D}_\gamma^l(\Pi^r)$ implies $p_T^r(u,(1-\chi)z_j^\pm)=0$. Together with (3.19) this allows us to write (3.26) in the form

$$\begin{aligned} &(\mathfrak{F}, z_j^\pm)_{\Pi^r} + (\mathfrak{G}, \mathfrak{Q}_T^r z_j^\pm)_{\partial\Pi^r} \\ &\quad = p_T^r\big(\sum_{h=1}^{M^r}\{a_h(\mathfrak{F},\mathfrak{G})\, z_h^+ + b_h(\mathfrak{F},\mathfrak{G})\, z_h^-\} + v, \chi z_j^\pm\big). \end{aligned}$$

Note that $p^r_T(v,\chi z_j^\pm) = 0$ as far as $v \in \mathcal{D}^l_\gamma(\Pi^r)$ and $\chi z_j^\pm \in \mathcal{D}^l_{-\gamma}(\Pi^r)$; see (3.11). Finally we have

$$\begin{aligned}
&(\mathfrak{F}, z_j^\pm)_{\Pi^r} + (\mathfrak{G}, \mathfrak{Q}^r_T z_j^\pm)_{\partial\Pi^r} \\
&\quad = -p^r_T(\sum_{h=1}^{M^r}\{a_h(\mathfrak{F},\mathfrak{G})\, z_h^+ + b_h(\mathfrak{F},\mathfrak{G})\, z_h^-\}, \chi z_j^\pm) \\
&\quad = -p^r_T(\sum_{h=1}^{M^r}\{a_h(\mathfrak{F},\mathfrak{G})\, \chi z_h^+ + b_h(\mathfrak{F},\mathfrak{G})\, \chi z_h^-\}, \chi z_j^\pm).
\end{aligned}$$

By applying Proposition 3.3, we complete the proof. □

Theorem 3.4 does not allow us to write a structure of $u \in \mathcal{D}^l_\beta(\Pi^r)$ with a remainder $v \in \mathcal{D}^l_\gamma(\Pi^r)$ if $\beta \neq -\gamma$. Further we correct this trouble.

Let α_ν, $\nu \in \mathbb{Z}$, be numbers such that every strip $\alpha_\nu \leq \operatorname{Im}\lambda < \operatorname{Im}\lambda_\nu$ is free of the spectrum of $\mathfrak{A}^r$. For sufficiently large T we set

$$w_\nu^{(\sigma,j)} = u_\nu^{(\sigma,j)} + \sum_{q=1}^{\infty}((A^r_{\alpha_\nu})^{-1}\Delta^r_T)^q u_\nu^{(\sigma,j)}, \tag{3.27}$$

where the functions $u_\nu^{(\sigma,j)}$ are given in (2.19) and satisfy the conditions (2.21)–(2.23). Repeating the arguments form the proof of Proposition 3.2 one can show that $w_\nu^{(\sigma,j)}$ solves the homogenous model problem (3.9). The functions $w_\nu^{(\sigma,j)}$ do not depend on the choice of α_ν; indeed, $(A^r_\alpha)^{-1}\{F,G\} = (A^r_\beta)^{-1}\{F,G\}$ provided that the strip $\alpha \leq \operatorname{Im}\lambda \leq \beta$ is free of the spectrum of the pencil $\mathfrak{A}^r$ and $\{F,G\} \in \mathcal{R}^l_\alpha(\Pi^r) \cap \mathcal{R}^l_\beta(\Pi^r)$ (see e.g. [3, Proposition 3.1.4]). From the relations $w_\nu^{(\sigma,j)} = u_\nu^{(\sigma,j)} \mod \mathcal{D}^l_{\alpha_\nu}(\Pi^r)$ and the formulas (2.19) it follows the linear independence of functions $w_\nu^{(\sigma,j)}$.

Lemma 3.5. *Let the assumptions of Theorem 3.4 be fulfilled and let $\lambda_{-M}, \dots, \lambda_M$ be all eigenvalues of $\mathfrak{A}^r$ from the strip $-\gamma < \operatorname{Im}\lambda < \gamma$. Then for sufficiently large T the relations*

$$w_\mu^{(\tau,p)} = \sum_{\nu=1}^{M}\{a_{\mu,\nu}^{(\tau,p)} z_\nu^+ + b_{\mu,\nu}^{(\tau,p)} z_\nu^-\}, \tag{3.28}$$

hold with the coefficients

$$a_{\mu,\nu}^{(\tau,p)} = ip^r_T(\chi w_\mu^{(\tau,p)}, \chi z_\nu^+), \qquad b_{\mu,\nu}^{(\tau,p)} = -ip^r_T(\chi w_\mu^{(\tau,p)}, \chi z_\nu^-), \tag{3.29}$$

where $\mu = -M, \dots, M$, $p = 1, \dots, J_\mu$, *and* $\tau = 0, \dots, \varkappa_{p\mu} - 1$; $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1$ *for* $t > 2$ *and* $\chi(t) = 0$ *for* $t < 1$.

Proof. Since $w_\mu^{(\tau,p)} \in \mathcal{D}^l_{\alpha_\mu}(\Pi^r)$, where $\gamma > \alpha_\mu \geq -\gamma$, we have $\chi w_\mu^{(\tau,p)} \in \mathcal{D}^l_{-\gamma}(\Pi^r)$ and $(\chi - 1) w_\mu^{(\tau,p)} \in \mathcal{D}^l_\gamma(\Pi^r)$. We put $\{\mathfrak{F}, \mathfrak{G}\} = -\{\mathfrak{L}^r_T, \mathfrak{B}^r_T\} \chi w_\mu^{(\tau,p)}$. It is clear that $\{\mathfrak{F}, \mathfrak{G}\} \in \mathcal{R}^l_\gamma(\Pi^r) \cap \mathcal{R}^l_{-\gamma}(\Pi^r)$ and $\{\mathfrak{L}^r_T, \mathfrak{B}^r_T\}(1-\chi) w_\mu^{(\tau,p)} = \{\mathfrak{F}, \mathfrak{G}\}$. By Proposition 3.1 and Theorem 3.4 we have

$$(1-\chi) w_\mu^{(\tau,p)} = \sum_{\nu=1}^{M} \{a_{\mu,\nu}^{(\tau,p)} z_\nu^+ + b_{\mu,\nu}^{(\tau,p)} z_\nu^-\} - \chi w_\mu^{(\tau,p)}.$$

This leads to (3.28). The equalities (3.29) are readily apparent from (3.28) and Proposition 3.3. $\square$

Proposition 3.6. *Let* $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1$ *for* $t \geq 2$ *and* $\chi(t) = 0$ *for* $t \leq 1$. *The functions* $w_\nu^{(\sigma,\nu)}$ *given in (3.27) satisfy the following conditions:*

$$p^r_T(\chi w_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = i\delta_{-\nu,\mu}\delta_{j,p}\delta_{\varkappa_{j\nu}-1-\sigma,\tau}, \quad |\nu| > \nu_0, |\mu| > \nu_0, \tag{3.30}$$

$$p^r_T(\chi w_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = \pm i\delta_{\nu,\mu}\delta_{j,p}\delta_{\varkappa_{j\nu}-1-\sigma,\tau}, \quad |\nu| \leq \nu_0, |\mu| \leq \nu_0, \tag{3.31}$$

$$p^r_T(\chi w_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = 0, \quad |\nu| \leq \nu_0, |\mu| > \nu_0. \tag{3.32}$$

In (3.31) the sign depends on ν *and* j *and coincides with the sign in (2.22). The conditions (3.30) – (3.32) do not depend on the choice of* χ.

Proof. First we prove that $p^r_T(\chi w_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = 0$ if $\operatorname{Im}(\lambda_\nu + \lambda_\mu) \neq 0$.

Let $\operatorname{Im}(\lambda_\nu + \lambda_\mu) > 0$. In this case one can choose α_ν and α_μ (see (3.27)) such that $\alpha_\nu + \alpha_\mu > 0$. Then for the functions $u := \chi w_\nu^{(\sigma,j)} \in \mathcal{D}^l_{\alpha_\nu}(\Pi^r)$ and $v := \chi w_\mu^{(\tau,p)} \in \mathcal{D}^l_{\alpha_\mu}(\Pi^r)$ the Green formula (3.2) holds. This implies $p^r_T(\chi w_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}) = 0$.

Let us consider the case $\operatorname{Im}(\lambda_\nu + \lambda_\mu) < 0$. One can choose β_ν and β_μ such that $\beta_\nu > \operatorname{Im} \lambda_\nu$, $\beta_\mu > \operatorname{Im} \lambda_\mu$, and $\beta_\nu + \beta_\mu < 0$. Then $(1-\chi) w_\nu^{(\sigma,j)} \in \mathcal{D}^l_{\beta_\nu}(\Pi^r)$, $(1-\chi) w_\mu^{(\tau,p)} \in \mathcal{D}^l_{\beta_\mu}(\Pi^r)$ and for $u := (1-\chi) w_\nu^{(\sigma,j)}$ and $v := (1-\chi) w_\mu^{(\tau,p)}$ the Green formula (3.2) holds. This implies $p^r_T((1-\chi) w_\nu^{(\sigma,j)}, (1-\chi) w_\mu^{(\tau,p)}) = 0$. Since the Green formula (3.2) holds for $u, v \in C_c^\infty(\overline{G})$ and $p^r_T(w_\nu^{(\sigma,j)}, w_\mu^{(\tau,p)}) = 0$, we get

$$p^r_T((1-\chi) w_\nu^{(\sigma,j)}, (1-\chi) w_\mu^{(\tau,p)}) = p^r_T((1-\chi) w_\nu^{(\sigma,j)}, w_\mu^{(\tau,p)}) = -p^r_T(\chi w_\nu^{(\sigma,j)}, \chi w_\mu^{(\tau,p)}).$$

Thus $p^r_T(\chi w^{(\sigma,j)}_\nu, \chi w^{(\tau,p)}_\mu) = 0$ if $\operatorname{Im}(\lambda_\nu + \lambda_\mu) \neq 0$.

Let $\operatorname{Im}(\lambda_\nu + \lambda_\mu) = 0$. Without loss of generality we can assume that $\operatorname{Im}\lambda_\nu \geq 0$. Then $\alpha_\mu < -\operatorname{Im}\lambda_\nu \leq 0$. We set $\{F, G\} = \{L^r, B^r\}(1-\chi)w^{(\sigma,j)}_\nu$. It is clear that $\{L^r, B^r\}(1-\chi)w^{(\sigma,j)}_\nu = \{\mathfrak{L}^r_T, \mathfrak{B}^r_T\}(1-\chi)w^{(\sigma,j)}_\nu \in \mathcal{R}^l_{-\alpha_\mu}(\Pi^r) \cap \mathcal{R}^l_{\alpha_\mu}(\Pi^r)$. Write down the asymptotic of the solution $(1-\chi)w^{(\sigma,j)}_\nu \in \mathcal{D}^l_{-\alpha_\mu}(\Pi^r)$ to the problem $\{L^r, B^r\}u = \{F, G\}$. We have

$$(1-\chi)w^{(\sigma,j)}_\nu = \sum_{h=-M}^{M} \sum_{s=1}^{J_h} \sum_{\delta=0}^{\varkappa_{sh}-1} c^{(\delta,s)}_h u^{(\delta,s)}_h \qquad \operatorname{mod}\ \mathcal{D}^l_{\alpha_\mu}(\Pi^r),$$

where $2M$ stands for the total algebraic multiplicity of all eigenvalues of $\mathfrak{A}^r$ in the strip $-\alpha_\mu > \operatorname{Im}\lambda > \alpha_\mu$; see e.g. [3, Proposition 3.1.4]. Note that $c^{(\sigma,j)}_\nu = 1$ because of the inclusion $(w^{(\sigma,j)}_\nu - u^{(\sigma,j)}_\nu) \in \mathcal{D}^l_{\alpha_\nu}(\Pi^r)$. Therefore,

$$\begin{aligned} -p^r_T(\chi w^{(\sigma,j)}_\nu, \chi w^{(\tau,p)}_\mu) &= p^r_T((1-\chi)w^{(\sigma,j)}_\nu, (1-\chi)w^{(\tau,p)}_\mu) \\ = q^r((1-\chi)w^{(\sigma,j)}_\nu, (1-\chi)w^{(\tau,p)}_\mu) &= q^r(\sum_{h=-M}^{M} \sum_{s=1}^{J_h} \sum_{\delta=0}^{\varkappa_{sh}-1} c^{(\delta,s)}_h u^{(\delta,s)}_h, (1-\chi)u^{(\tau,p)}_\mu) \\ &= -q^r(\sum_{h=-M}^{M} \sum_{s=1}^{J_h} \sum_{\delta=0}^{\varkappa_{sh}-1} c^{(\delta,s)}_h \chi u^{(\delta,s)}_h, \chi u^{(\tau,p)}_\mu) \end{aligned}$$

(in the next-to-last equality we used that $q^r(u, v) = 0$ for $u \in \mathcal{D}^l_{-\alpha_\mu}(\Pi^r)$ and $v \in \mathcal{D}^l_{\alpha_\mu}(\Pi^r)$). Taking into account the equality $c^{(\sigma,j)}_\nu = 1$ and the relations (2.21)–(2.23), we complete the proof. □

Theorem 3.7. *Assume that $\mathcal{L}$ and $\mathcal{R}$ stabilize in Π^r_+. We also suppose that $\beta > \alpha$ and the lines $\mathbb{R} + i\alpha$ and $\mathbb{R} + i\beta$ contain no eigenvalues of the pencil $\mathfrak{A}^r$. Let $\lambda_K, \dots, \lambda_M$ be all eigenvalues of $\mathfrak{A}^r$ from the strip $\alpha < \operatorname{Im}\lambda < \beta$ and $\{\mathfrak{F}, \mathfrak{G}\} \in \mathcal{R}^l_\alpha(\Pi^r) \cap R^l_\beta(\Pi^r)$. Then for sufficiently large T the following assertions hold.*
(i) *A solution $u \in \mathcal{D}^l_\alpha(\Pi^r)$ to the model problem (3.9) admits the representation*

$$u = \sum_{\nu=K}^{M} \sum_{j=1}^{J_\nu} \sum_{\sigma=0}^{\varkappa_{j\nu}-1} d^{(\sigma,j)}_\nu(\mathfrak{F}, \mathfrak{G})\, w^{(\sigma,j)}_\nu + v, \tag{3.33}$$

where v is a solution to the same problem in $\mathcal{D}^l_\beta(\Pi^r)$.

(ii) *The coefficients* $d_\nu^{(\sigma,j)}(\mathfrak{F},\mathfrak{G})$ *in (3.33) can be found by the formulas*

$$\begin{aligned} d_\nu^{(\sigma,j)}(\mathfrak{F},\mathfrak{G}) = i\{(\mathfrak{F}, &w_{-\nu}^{(\varkappa_{j\nu}-\sigma-1,j)})_{\Pi^r} \\ &+ (\mathfrak{G}, \mathfrak{Q}_T^r w_{-\nu}^{(\varkappa_{j\nu}-\sigma-1,j)})_{\partial\Pi^r}\}, \quad |\nu| > \nu_0, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} d_\nu^{(\sigma,j)}(\mathfrak{F},\mathfrak{G}) = \pm i\{(\mathfrak{F}, &w_{\nu}^{(\varkappa_{j\nu}-\sigma-1,j)})_{\Pi^r} \\ &+ (\mathfrak{G}, \mathfrak{Q}_T^r w_{\nu}^{(\varkappa_{j\nu}-\sigma-1,j)})_{\partial\Pi^r}\}, \quad |\nu| \le \nu_0. \end{aligned} \tag{3.35}$$

The sign in (3.35) is the same as in (2.22).

Proof. Let $\gamma = \max\{-\alpha, \beta\}$. We again use the cut-off function $\chi \in C_\infty^c(\overline{G})$, $\chi(t) = 1$ for $t \ge 2$ and $\chi(t) = 0$ for $t \le 1$.

Let us first consider the case $\gamma = -\alpha$. Let $v \in \mathcal{D}_\beta^l(\Pi^r)$ satisfy the model problem (3.9). We set $\{F, G\} := \{\mathfrak{L}_T^r, \mathfrak{B}_T^r\}(1-\chi)v \in \mathcal{R}_{-\gamma}^l(\Pi^r) \cap \mathcal{R}_\gamma^l(\Pi^r)$. By Theorem 3.4 we have

$$y = \sum_{j=1}^{-K} \{a_j z_j^+ + b_j z_j^-\} + (1-\chi)v, \tag{3.36}$$

where $y \in \mathcal{D}_{-\gamma}^l(\Pi^r)$. Due to the linear independence of $w_\nu^{(\sigma,j)}$ and (3.28) the waves $z_\nu^\pm$ can be expressed in terms of $w_\nu^{(\sigma,j)}$. From (3.36) we get

$$u = y + \chi v = \sum_{j=1}^{-K} \{a_j z_j^+ + b_j z_j^-\} + v = \sum_{\nu=K}^{-K} \sum_{j=1}^{J_\nu} \sum_{\sigma=0}^{\varkappa_{j\nu}-1} d_\nu^{(\sigma,j)}(F,G)\, w_\nu^{(\sigma,j)} + v, \tag{3.37}$$

where $u = (\chi v + y) \in \mathcal{D}_\alpha^l(\Pi^r)$. Since $(1-\chi)(u-v) \in \mathcal{D}_\beta^l(\Pi^r)$ and the function $(1-\chi)w_\nu^{(\sigma,j)}$ is in $\mathcal{D}_\beta^l(\Pi^r)$ only if $\nu \ge K$, we have $d_\nu^{(\sigma,j)}(F,G) = 0$ for $\nu = M+1, \dots, -K$. In the case $-\alpha \ge \beta$ the representation (3.33) is proved.

Consider the case $-\alpha < \beta$. Then $\gamma = \beta$. Applying Theorem 3.4, we get the representation

$$\chi u = \sum_{j=1}^{M} \{a_j z_j^+ + b_j z_j^-\} + y, \quad y \in \mathcal{D}_\gamma^l(\Pi^r),$$

for the solution $\chi u \in \mathcal{D}^l_{-\gamma}(\Pi^r)$ to the model problem (3.9) with right-hand side $\{F, G\} := \{\mathfrak{L}^r_T, \mathfrak{B}^r_T\}\chi u \in \mathcal{R}^l_{-\gamma}(\Pi^r) \cap \mathcal{R}^l_{\gamma}(\Pi^r)$. Therefore,

$$u = \sum_{\nu=-M}^{M} \sum_{j=1}^{J_\nu} \sum_{\sigma=0}^{\varkappa_{j\nu}-1} d_\nu^{(\sigma,j)}(F, G)\, w_\nu^{(\sigma,j)} + v,$$

where $v = (y+(1-\chi)u) \in \mathcal{D}^l_\beta(\Pi^r)$. Owing to the inclusion $\chi(u-v) \in \mathcal{D}^l_\alpha(\Pi^r)$, we have $d_\nu^{(\sigma,j)}(F, G) = 0$ for $\nu = -M, \dots, K-1$. The assertion (i) of the theorem is proved.

Using Proposition 3.6 and the representation (3.33) one can prove the formulas (3.34) and (3.35) for the coefficients; see the proof of (3.20) in Theorem 3.4. □

3.3 The structure of solutions to the problem (2.1)

Let us define the spaces $\mathcal{D}^\ell_\gamma(G)$ and $\mathcal{R}^\ell_\gamma(G)$ by the equalities (2.16) with Π^r replaced by G; the space $W_\gamma(G)$ is endowed with the norm $\|e_\gamma \cdot; H^\ell(G)\|$, where e_γ is smooth positive function in $\overline{G}$ such that $e_\gamma(y^r, t^r) = \exp \gamma t^r$ for $(y^r, t^r) \in \bar{\Pi}^r$.

Assume that the operators $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ stabilize in $\Pi^1_+, \dots, \Pi^N_+$. As was shown the stabilization in Π^r_+ implies (3.8). Thus the operator

$$\mathcal{A}(\gamma) = \{\mathcal{L}, \mathcal{B}\} : \mathcal{D}^\ell_\gamma(G) \to \mathcal{R}^\ell_\gamma(G) \tag{3.38}$$

of the problem (2.1) is continuous.

Proposition 3.1 and the well known results of the local theory of elliptic boundary value problems enable one to prove the following proposition in the standard way. The proof is omitted.

Proposition 3.8. *Let the operators $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ stabilize in $\Pi^1_+, \dots, \Pi^N_+$ and let the line $\mathbb{R} + i\gamma$ be free of the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. Assume that T is sufficiently large. Then the operator (3.38) of the problem (2.1) is Fredholm.*

Let us introduce the space of waves $\mathcal{W}_\gamma(G)$. Suppose that the assumptions of Proposition (3.8) are fulfilled. We extend the functions $\chi z_j^\pm$, $j = 1, \dots, M^r$, from the semicylinder Π^r_+ to the domain G by zero and set

$$v_h^\pm := \chi z_j^\pm, \quad h = j + \sum_{p=1}^{r-1} M^p,\ j = 1, \dots, M^r,\ r = 1, \dots, N; \tag{3.39}$$

here $z_j^\pm$ are defined by (3.10), the cut-off function χ is the same as in Proposition 3.3. Let $\mathcal{W}_\gamma(G)$ be the space spanned by functions of the form $v_h^\pm + v$, where v is a function in $\mathcal{D}_\gamma^\ell(G)$, and $h = 1, \dots, M$, $M = \sum_{r=1}^N M^r$. It is clear that $\mathcal{W}_\gamma(G) \subset \mathcal{D}_{-\gamma}^\ell(\Pi^r)$. Note that the elements of $\mathcal{W}_\gamma(G)$ do not necessary satisfy the homogeneous problem (2.1).

Denote

$$q(u, v) = (\mathcal{L}u, v)_G + (\mathcal{B}u, \mathcal{Q}v)_{\partial G} - (u, \mathcal{L}v)_G - (\mathcal{Q}u, \mathcal{B}v)_{\partial G}. \tag{3.40}$$

The quantity $iq(u, u)$ represents the total energy flow transferred by the wave $u \in \mathcal{W}_\gamma(G)$ through the infinitely distant cross-sections $\Omega^1, \dots, \Omega^N$ of the cylindrical ends of the domain G. It is easy to see that $iq(u, u) = 0$ for an exponentially decreasing function $u \in \mathcal{D}_\varepsilon^l(G)$, $\varepsilon > 0$.

Lemma 3.9. *Under the circumstances of Proposition 3.8 the waves $v_j^\pm$, $j = 1, \dots, M$, given by (3.39) satisfy the conditions*

$$q(v_h^\pm, v_j^\pm) = \mp i\delta_{h,j}, \quad q(v_h^\pm, v_j^\mp) = 0, \quad j, h = 1, \dots, M. \tag{3.41}$$

Thus $v_1^+, \dots, v_M^+$ are incoming waves and $v_1^-, \dots, v_M^-$ are outgoing waves for the problem (2.1).

Proof. By Proposition 3.3 the conditions (3.15) are valid. Due to the Green formula (3.2) we can replace in (3.15) the cut-off function χ by a cut-off function $\zeta_T \in C^\infty(\mathbb{R})$, $\zeta_T(t) = 1$ for $t > 3T$ and $\zeta_T(t) = 0$ for $t < 2T$. Recall that the operator $\{\mathcal{L}, \mathcal{B}\}$ coincides with $\{\mathfrak{L}_T^r, \mathfrak{B}_T^r\}$ on the set $\{(y^r, t^r) \in \overline{\Pi}^r : t^r > T + 3\}$; see section 3.1. If $v_j^\pm$ and $v_h^\pm$ are related to different semicylinders Π_+^r and Π_+^s then those supports do not overlap. We have

$$q(\zeta_T v_h^\pm, \zeta_T v_j^\pm) = \mp i\delta_{h,j}, \quad q(\zeta_T v_h^\pm, \zeta_T v_j^\mp) = 0, \quad j, h = 1, \dots, M.$$

Owing to the Green formula (2.5) the cut-off function ζ_T can be omitted. □

Theorem 3.10. *Let $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ stabilize in $\Pi_+^1, \dots, \Pi_+^N$ and let the line $\mathbb{R} + i\gamma$ be free of the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. Then for a solution $u \in \mathcal{D}_{-\gamma}^\ell(G)$ to the problem (2.1) with right-hand side $\{\mathcal{F}, \mathcal{G}\} \in \mathcal{R}_\gamma^\ell(G)$ the inclusion*

$$u - \sum_{j=1}^M \{a_j v_j^+ + b_j v_j^-\} \in \mathcal{D}_\gamma^\ell(G) \tag{3.42}$$

holds. Here

$$a_j = iq(u, v_j^+), \quad b_j = -iq(u, v_j^-), \quad j = 1, \dots, M, \tag{3.43}$$

the waves $v_j^\pm$ are defined by (3.39) and (3.10), where T is sufficiently large.

Proof. Let $\zeta_T \in C^\infty(\overline{G})$, $\zeta_T(t) = 1$ for $t > 3T$ and $\zeta_T(t) = 0$ for $t < 2T$. Denote by ζ_T^r the cut-off function such that ζ_T^r coincides with ζ_T inside $\overline{\Pi}_+^r$ and vanishes on the remaining part of $\overline{G}$. Theorem 3.4 implies the representations of the form (3.19) for the solutions $\zeta_T^r u \in \mathcal{D}^l_{-\gamma}(\Pi^r)$ to the problems (3.9) with the right-hand sides $\{\mathfrak{F}^r, \mathfrak{G}^r\} := \{\mathfrak{L}_T^r, \mathfrak{B}_T^r\}\zeta_T^r u$, $r = 1, \dots, N$. To prove (3.42) it remains to note that $\zeta_T \chi = \zeta_T$ and $(1 - \sum_{r=1}^N \zeta_T^r)u \in \mathcal{D}^\ell_\gamma(G)$.

The equalities (3.43) directly follow from (3.42) and Lemma 3.9. $\square$

Theorem 3.11. *Assume that $\mathcal{L}$ and $\mathcal{R}$ stabilize in Π_+^r. We also suppose that $\beta > \alpha$ and the lines $\mathbb{R} + i\alpha$ and $\mathbb{R} + i\beta$ contain no eigenvalues of the pencil $\mathfrak{A}^r$. Let $\lambda_K, \dots, \lambda_M$ be all eigenvalues of $\mathfrak{A}^r$ from the strip $\alpha < \operatorname{Im}\lambda < \beta$ and let $\eta\{\mathfrak{F}, \mathfrak{G}\} \in \mathcal{R}^l_\beta(G)$, where $\eta \in C^\infty(\overline{G})$, $\operatorname{supp}\eta \in \overline{\Pi}_+^r$ and $\eta = 1$ on the set $\{(y,t) \in \overline{\Pi}_+^r, t > 3\}$. If u is a solution to the problem (2.1) such that $\eta u \in \mathcal{D}^l_\alpha(G)$ then inside Π_+^r the representation*

$$u = \sum_{\nu=K}^{M} \sum_{j=1}^{J_\nu} \sum_{\sigma=0}^{\varkappa_{j\nu}-1} c_\nu^{(\sigma,j)}(\mathfrak{F}, \mathfrak{G})\, w_\nu^{(\sigma,j)} + v \tag{3.44}$$

holds, where $\eta v \in \mathcal{D}^l_\beta(G)$ and

$$c_\nu^{(\sigma,j)}(\mathfrak{F}, \mathfrak{G}) = iq(u, \eta w_{-\nu}^{(\varkappa_{j\nu}-\sigma-1,j)}), \quad |\nu| > \nu_0,$$

$$c_\nu^{(\sigma,j)}(\mathfrak{F}, \mathfrak{G}) = \pm iq(u, \eta w_\nu^{(\varkappa_{j\nu}-\sigma-1,j)}), \quad |\nu| \le \nu_0.$$

The sign in the last formula is the same as in (2.22).

Proof. The assertion follows from the item (i) of Theorem 3.7 and Proposition 3.6. $\square$

4 Corollaries of Theorems 3.10 and 3.11

4.1 Index properties, Scattering matrices, An existence criterion of exponentially decaying solutions

Proposition 4.1. *Let the assumptions of Theorem 3.11 be fulfilled. Then the indexes of operators $\mathcal{A}(\alpha)$ and $\mathcal{A}(\beta)$ are connected by the relation*

$$\operatorname{Ind} \mathcal{A}(\alpha) - \operatorname{Ind} \mathcal{A}(\beta) = \varkappa,$$

where $\varkappa$ is the total algebraic multiplicity of all eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ in the strip $\{\lambda \in \mathbb{C} : \alpha < \operatorname{Im} \lambda < \beta\}$.

The assertion of this proposition follows from the structure (3.44) of solution to the problem (2.1); see [3, Section 4.3]. Using Proposition 4.1 and the formal self-adjointness of $\{\mathcal{L}, \mathcal{B}\}$ one can prove the following proposition; see [3, Section 5.1.3].

Proposition 4.2. *Let the assumptions of Theorem 3.10 be fulfilled. Then*

$$\dim \ker \mathcal{A}(-\gamma) - \dim \ker \mathcal{A}(\gamma) = \dim \operatorname{coker} \mathcal{A}(\gamma) - \dim \operatorname{coker} \mathcal{A}(-\gamma) = M,$$

where $2M$ is the total algebraic multiplicity of all eigenvalues of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| < \gamma\}$.

The next proposition is a corollary of the formulas (3.43) for the coefficients in the structure (3.42) of solution; see [3, Propositions 5.3.3, 5.3.4].

Proposition 4.3. *Let the assumptions of Theorem 3.10 be fulfilled. Then there exist bases $Z_1, \dots, Z_M$ and $X_1, \dots, X_M$ in the space $\ker \mathcal{A}(-\gamma)$ modulo $\mathcal{D}^\ell_\gamma W(G)$ such that*

$$Z_k - \left(v_k^+ + \sum_{j=1}^{M} \mathfrak{T}_{kj} v_j^- \right) \in \mathcal{D}^\ell_\gamma W(G), \quad k = 1, \dots, M, \tag{4.1}$$

$$X_k - \left(v_k^- + \sum_{j=1}^{M} \mathfrak{S}_{kj} v_j^+ \right) \in \mathcal{D}^\ell_\gamma W(G), \quad k = 1, \dots, M, \tag{4.2}$$

where the scattering matrices $\mathfrak{T} \equiv \|\mathfrak{T}_{kj}\|$ and $\mathfrak{S} \equiv \|\mathfrak{S}_{kj}\|$ of sizes $M \times M$ are unitary, i.e. $\mathfrak{T}^* = \mathfrak{T}^{-1}$ and $\mathfrak{S}^* = \mathfrak{S}^{-1}$; moreover, $\mathfrak{S} = \mathfrak{T}^{-1}$.

Before formulating an existence criterion of exponentially decaying solutions to the homogeneous problem (2.1) we need to construct a special basis $\{v_j^\pm\}_{j=1}^{M'}$ modulo $\mathcal{D}^l_\beta(G)$ in the space of waves $\mathcal{W}_\beta(G)$, $\beta > \gamma$.

Lemma 4.4. *Let $0 < \gamma < \beta$, and let $\{v_j^\pm\}_{j=1}^{M}$ be a basis in the space of waves $\mathcal{W}_\gamma(G)$ modulo $\mathcal{D}^l_\gamma(G)$ subjected to (3.41). The set $\{v_j^\pm\}_{j=1}^{M}$ can be supplemented to a basis $\{v_j^\pm\}_{j=1}^{M'}$ in $\mathcal{W}_\beta(G)$ modulo $\mathcal{D}^l_\beta(G)$ so that $v_s^+ + v_s^- \in \mathcal{D}^l_\gamma W(G)$ for $s = M+1, \dots, M'$ and the relations* (3.41) *hold for $h, j = 1, \dots, M'$.*

Proof. In fact the waves $v_s^\pm$, $s = M+1, \dots, M'$ can be constructed in the same way as the waves $u_s^\pm$ (see e.g. [11]), one has to use the functions (3.27) instead of functions (2.19).

For simplicity of description we suppose that domain G has only one cylindrical end Π^1_+. Assume that the Jordan chains of the pencil $\mathfrak{A}^1$ are chosen such that the conditions (2.21)–(2.23) for the functions (2.19) are valid. To the every Jordan chain $\{\varphi_\nu^{(0,j)}, \dots, \varphi_\nu^{(\varkappa_{j\nu}-1,j)}\}$ there correspond the functions $w_\nu^{(0,j)}, \dots, w_\nu^{(\varkappa_{j\nu}-1,j)}$ given by (3.27). By Proposition 3.6 the conditions (3.30)–(3.32) are valid. With every eigenvalue λ_ν of $\mathfrak{A}^1$ such that $\gamma < \operatorname{Im}\lambda_\nu < \beta$ we associate the functions

$$w_{\nu,\pm}^{(\sigma,j)} = 2^{-1/2}\chi(w_\nu^{(\sigma,j)} \mp w_{-\nu}^{(\varkappa_{j\nu}-\sigma-1,j)}), \quad j = 1, \dots, J_\nu, \tau = 0, 1, \dots, \varkappa_{j\nu} - 1, \tag{4.3}$$

where $J_\nu = \dim\ker\mathfrak{A}^1(\lambda_\nu)$, χ is the same as in (3.39). Then owing to (3.30) and (3.32) we have $p_T^1(w_{\nu,\pm}^{(\sigma,j)}, w_{\mu,\pm}^{(\tau,p)}) = \mp i\delta_{\nu,\mu}\delta_{\sigma,\tau}\delta_{j,p}$; the wave $w_{\nu,+}^{(\sigma,j)}$ is incoming and the wave $w_{\nu,-}^{(\sigma,j)}$ is outgoing. Due to the linear independence of the functions $w_\nu^{(\sigma,j)}$ and Lemma 3.5, the elements of the basis $\{v_j^\pm\}_{j=1}^{M}$ can be expressed in terms of functions $\chi w_\nu^{(\sigma,j)}$ corresponding to the eigenvalues of $\mathfrak{A}^1$ in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$. Together with (3.32) this implies $p_T^1(w_{\nu,\pm}^{(\sigma,j)}, v_s^\pm) = 0$ for $s = 1, \dots, M$. It remains to note that $w_{\nu,+}^{(\sigma,j)} + w_{\nu,-}^{(\sigma,j)} = 2\chi w_\nu^{(\sigma,j)} \in \mathcal{D}^l_\gamma(G)$. As $v_j^\pm$, $j = M+1, \dots, M'$, we can take the waves $w_{\nu,\pm}^{(\sigma,j)}$. □

Proposition 4.5. *Let $0 < \gamma < \beta$, and let the lines $\mathbb{R} + i\gamma$ and $\mathbb{R} + i\beta$ be free of the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. Denote by $\mathfrak{S} = \mathfrak{S}(\beta)$ the scattering matrix corresponding to the basis $\{v_j^\pm\}_{j=1}^{M'}$ from Lemma 4.4. Then*

$$\dim\ker\mathcal{A}(\gamma) - \dim\ker\mathcal{A}(\beta) = \dim\ker(\mathfrak{S}^{2,2} - I),$$

where $\mathfrak{S}^{2,2}$ *is* $(M'-M)\times(M'-M)$*-block of the* $M'\times M'$*-matrix* $\mathfrak{S} = \|S^{k,\ell}(\gamma')\|_{k,\ell=1,2}$.

The proof is similar to the proof of Theorem 3.3 from [11].

4.2 Problem with radiation conditions

As before we suppose that $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ are stabilizing in $\Pi^1_+, \dots, \Pi^N_+$ and the line $\mathbb{R}+i\gamma$ is free of the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. Denote by $\mathcal{W}_{out}(G)$ the linear span of the outgoing waves $v^-_1, \dots, v^-_M$ and consider the restriction A of $\mathcal{A}(-\gamma)$ to the space $\mathfrak{D}_{out}(G) = \mathcal{W}_{out}(G)[\dot{+}]\mathcal{D}^l_\gamma(G)$, where by $[\dot{+}]$ we denote the orthogonal with respect to the form (3.40) direct sum. The mapping $A : \mathfrak{D}_{out}(G) \to \mathcal{R}^l_\gamma(G)$ is continuous.

Proposition 4.6. *Let* $z_1, \dots, z_d$ *be a basis of* $\ker \mathcal{A}(\gamma)$*, and let* $\{f, g\} \in \mathcal{R}^l_\gamma(G)$*,* $(f, z_j)_G + (g, \mathcal{Q}z_j)_{\partial G} = 0$*,* $j = 1, \dots, d$*.*
(i) *There exists a unique up to an arbitrary element of* $\ker \mathcal{A}(\gamma)$ *solution* $u \in \mathfrak{D}_{out}(G)$ *to the problem (2.1).*
(ii) *The inclusion*

$$v \equiv u - b_1 v^-_1 - b_2 v^-_2 - \dots - b_M v^-_M \in \mathcal{D}^\ell_\gamma W(G)$$

holds with the coefficients

$$b_j = -i(f, X_j)_G - i(g, \mathcal{Q}X_j)_{\partial G}, \quad j = 1, \dots, M,$$

where $X_1, \dots, X_M$ *are elements of* $\ker \mathcal{A}(-\gamma)$ *subjected to (4.2).*
(iii) *The solution* u *satisfies the inequality*

$$\begin{aligned} &\|v; \mathcal{D}^\ell_\gamma W(G)\| + |b_1| + |b_2| + \dots + |b_M| \\ &\qquad \leqslant C(\|\{f, g\}; \mathcal{R}^\ell_\gamma W(G)\| + \|e_\gamma v; L_2(G)\|). \end{aligned} \tag{4.4}$$

4. *The solution* u *subjected to the additional conditions* $(u, z_j)_G = 0$*,* $j = 1, \dots, d$*, is unique and satisfies the estimate* (4.4) *with the right-hand side replaced by* $\|\{f, g\}; \mathcal{R}^\ell_\gamma W(G)\|$*.*

This proposition justifies the statement of the problem (2.1) with intrinsic radiation conditions (only outgoing “waves” occur in asymptotic formulas for solutions). Up to obvious changes the proof repeats the proof of Theorem 5.3.5 from [3]. The next two propositions describe the statement of the problem with other radiation conditions. For the proofs we refer to [3, Theorems 5.5.5, 5.5.6].

Proposition 4.7. *Let $\eta_1, \dots, \eta_d$ be a basis of* $\ker \mathcal{A}(\gamma)$*, and let the right-hand side $\{f, g\} \in \mathcal{R}^\ell_\gamma W(G)$ satisfy the orthogonality conditions $(f, \eta_j)_G + (g, \mathcal{Q}\eta_j)_{\partial G} = 0$, $j = 1, \dots, d$. We assume that for the space* $\ker \mathcal{A}(-\gamma)$ *one can choose a basis $V_1, \dots, V_M$ modulo $\mathcal{D}^l_\gamma W(G)$ that compatible with the basis $u_1, \dots, u_{2M}$ for the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ in the following sense*:

$$q(u_j, V_k) = -i\delta_{kj}, \quad k, j = 1, \dots, M. \tag{4.5}$$

Then the following assertions hold.

1. *There exists a unique up to an arbitrary element of* $\ker \mathcal{A}(\gamma)$ *solution $u \in \mathfrak{h}[\dotplus]\mathcal{D}^\ell_\gamma W(G)$ to the problem* (2.1)*, where $\mathfrak{h}$ is the linear span of the functions $u_1, \dots, u_M$.*

2. *The following inclusion holds*:

$$v \equiv u - b_1 u_1 - b_2 u_2 - \dots - b_M u_M \in \mathcal{D}^\ell_\gamma W(G),$$

where $b_j = i(f, V_j)_G + i(g, \mathcal{Q}V_j)_{\partial G}$, $j = 1, \dots, M$.

3. *The solution u satisfies the inequality (4.4).*

4. *The solution u subjectd to the additional conditions $(u, \eta_j)_G = 0$, $j = 1, \dots, d$, is unique and satisfies the estimate* (4.4) *with the right-hand side replaced by $\|\{f, g\}; \mathcal{R}^\ell_\gamma W(G)\|$.*

By Proposition 4.7, to enumerate all possible radiation conditions is the same that to enumerate all the bases $u_1, \dots, u_{2M}$ for the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ and bases $V_1, \dots, V_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ for the subspace $\ker \mathcal{A}_*(-\gamma)$ compatible in the sense of (4.5).

Proposition 4.8. *Let $\mathcal{W}_\gamma(G)$ be the space of waves and let the waves $v_j^\pm$, $j = 1, \dots, M$, form a basis of the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ subjected to (3.41). Denote by $\{X_1, \dots, X_M\}$ a set of solutions to the homogeneous problem $\{\mathcal{L}, \mathcal{B}\}u = 0$ satisfying the inclusions* (4.2)*. Then the following assertions hold.*

1. *If R is arbitrary and S is an invertible operator in $\mathbb{C}^M$, then*

$$\begin{aligned} V_k &= \sum_{m=1}^{M} \overline{(S^{-1})_{mk}} X_m, \\ u_j &= \sum_{m=1}^{M} \left(S_{jm} u_m^- + \sum_{p=1}^{M} R_{jp} \left\{ u_p^+ + \sum_{i=1}^{M} \mathfrak{S}^*_{pi} u_i^- \right\} \right), \end{aligned} \tag{4.6}$$

where $j, k = 1, \ldots, M$, satisfy the condition (4.5).

2. *If a basis $V_1, \ldots, V_M$ modulo $\mathcal{D}^\ell_\gamma W(G)$ for the space* $\ker \mathcal{A}(-\gamma)$ *and a basis $u_1, \ldots, u_{2M}$ for the quotient space $\mathcal{W}_\gamma(G)/\mathcal{D}^\ell_\gamma W(G)$ satisfy* (4.5)*, then there exist operators R and S such that the equalities* (4.6) *hold.*

4.3 The extensions of the symmetric operator

The schemes for the proofs of propositions listed in this section can be found in [3, Section 5.5], the changes in the proofs consist in usage of Theorem 3.10 instead of asymptotic representations.

Here we assume that the elliptic system $\{\mathcal{L}, \mathcal{B}\}$ is homogeneous. In other words, $\tau_1 = \tau_2 = \ldots = \tau_k \equiv \tau$ and $\mathcal{D}^\ell_\gamma W(G) := \prod_{i=1}^k W^{2\tau}_\gamma(G)$. With the problem (2.1) we associate an operator $\mathcal{M}$ with the domain

$$\mathcal{D}(\mathcal{M}) = \{u \in \mathcal{D}^\ell_\gamma W(G) : \mathcal{B}(x, D_x)u(x) = 0, x \in \partial G\}$$

that acts in the Hilbert space

$$L_2(G; e_{-\gamma}) \equiv \prod_{i=1}^{k} W^0_{-\gamma}(G)$$

by the formula

$$(\mathcal{M}u)(x) = e_\gamma(x)^2 \mathcal{L}(x, D_x)u(x).$$

We denote by $(\cdot, \cdot)_{-\gamma}$ the inner product

$$(u, v)_{-\gamma} = \int_G e_{-\gamma}(x)^2 u(x)\overline{v(x)}\, dx$$

in the space $L_2(G; e_{-\gamma})$.

Proposition 4.9. *Suppose that $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ are stabilizing and the line $\mathbb{R} + i\gamma$ is free of the spectrum of the pencils $\mathfrak{A}^1, \ldots, \mathfrak{A}^N$. Then the operator $\mathcal{M}$ is closed and symmetric. For the* $\operatorname{coker} \mathcal{M}$ *one can choose a basis $X_1, \ldots, X_M$ modulo $\mathcal{D}^l_\gamma W(G)$ such that the inclusions* (4.2) *hold.*

Proposition 4.10. *Let $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ stabilize and let the line $\mathbb{R}+i\gamma$ be free of the spectrum of the pencils $\mathfrak{A}^1, \ldots, \mathfrak{A}^N$. The operator $\mathcal{M}^*$, adjoint to $\mathcal{M}$ in the space $L_2(G; e_{-\gamma})$, is defined on the set*

$$\mathcal{D}(\mathcal{M}^*) = \{v \in \mathcal{W}_\gamma(G) : \mathcal{B}(x, D_x)v(x) = 0, x \in \partial G\}$$

and acts by the formula $(\mathcal{M}^*u)(x) = e_\gamma(x)^2 \mathcal{L}(x, D_x)u(x)$.

We extend the operator $\mathcal{M}$ by adjoining representatives of waves to the domain. Let $u \in \mathcal{D}(\mathcal{M}^*)$. Then u is a solution of the problem (2.1) with the right-hand side $\{f, g\} = \{\mathcal{L}u, 0\} \in L_2(G; e_\gamma)$. In accordance with propositions 4.7 and 4.3 the function u has the form

$$u = u^0 + \sum_{k=1}^{d} c_k \eta_k + \sum_{j=1}^{M} d_j X_j,$$

where u^0 is such that $u^0 - b_1 v_1^- - \dots - b_M v_M^- \in \mathcal{D}_\gamma^l(G)$. With every function $u \in \mathcal{D}(\mathcal{M}^*)$ we associate vectors $d = (d_1, \dots, d_M)$ and $b = (b_1, \dots, b_M)$ of the coefficients.

Proposition 4.11. *Suppose that $\mathcal{L}(x, D_x)$ and $\mathcal{R}(x, D_x)$ are stabilizing and the line $\mathbb{R} + i\gamma$ is free of the spectrum of the pencils $\mathfrak{A}^1, \dots, \mathfrak{A}^N$. Let $\mathbb{C}^M = \mathbb{N}_+ (+) \mathbb{N}_0 (+) \mathbb{N}_-$ be an orthogonal sum of subspaces and let K be a self-adjoint operator in $\mathbb{N}_0$. The operator $\mathbb{M}$ is a self-adjoint extension of $\mathcal{M}$ if and only if $\mathbb{M}$ is defined on the set*

$$\begin{aligned} \mathcal{D}(\mathbb{M}) = \{u \in \mathcal{D}(\mathcal{M}^*) : c = \zeta_+ + \zeta_0, \\ a = \zeta_- - (iK + I)\zeta_0/2 - \zeta_+/2, \zeta_\pm \in \mathbb{N}_\pm, \zeta_0 \in \mathbb{N}_0\} \end{aligned}$$

and acts by the formula $(\mathbb{M}u)(x) = e_\gamma(x)^2 \mathcal{L}(x, D_x) u(x)$.

References

[1] V. Maz'ya, B. Plamenevskii, " Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary," Math. Nachr. 81, 25–82. English translation: Amer. Math. Soc. Transl. 123 (1984), 1–56.

[2] V. Kozlov, V. Maz'ya, "Differential equations with operator coefficients with applications to boundary value problems for partial differential equations," Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1999.

[3] S.A. Nazarov, B.A. Plamenevsky, "Elliptic problems in domains with piecewise smooth boundaries," Walter de Gruyter, Berlin, 1994.

[4] V. O. Kalvine, P. Neittaanmäki, and B. A. Plamenevskii, *On accumulations of the point spectra of elliptic problems in domains with cylindrical ends* [in Russian], *Dokl. Akad. Nauk*, **394** (2004); English transl.: *Dokl. Math.*, **69** (2004), 92–94.

[5] Ja.A. Roĭtberg, Z.G. Sheftel', " A theorem of homeomorphisms for elliptic systems and its applications," Mat. Sb. Vol. 78 (1969), 446–472; Math. USSR-Sb. Vol. 7 (1969), 439–465.

[6] J.L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I,* Springer-Verlag, Berlin Heidelberg New York, 1972.

[7] N. Aronszajn, A. N. Milgram, " Differential operators on Riemannian manifolds", Rend. Circ. Mat. Palermo (2) 2 (1953), 266–325.

[8] M. Schechter, " General boundary value problems for elliptic partial differential equations," Comm. Pure Appl. Math. 12 (1959), 457–486.

[9] Ju. M. Berezanskii, S.G. Krein, and Ja. A. Roitberg, " A theorem on homeomorphisms and local increase of smoothness up to the boundery for solutions of elliptic equations," Dokl. Akad. Nauk SSSR 148 (1963), 745–748. English translation: Soviet Math. Dokl. 4 (1963), 152–155.

[10] I.C. Gohberg, E.I. Sigal, " An operator generalization of the logarithmic residue theorem and the theorem of Rouché," Mat. Sb. Vol. 84 (1971), 607–629; Math. USSR-Sb. Vol. 13 (1971), 603–625.

[11] I.V. Kamotsky, S.A. Nazarov, " An augmented scattering matrix and exponentially vanishing solutions of an elliptic problem in a cylindrical domain," Zap. Nauchn. Sem. St. Petersburg Otdel. Mat. Inst. Steklov (POMI) Vol. 264 (2000), 66–82, 322–323.